University College London

Portfolio Choice with Independent Components:
Applications in Infrastructure Investment

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2013

A thesis submitted to University College London in partial fulfilment of the requirements for the degree of Doctorate of Philosophy in Financial Economics

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I, Maximilian A.M. Vermorken confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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19 August 2013

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ABSTRACT

One of the principal questions in financial economics and applied finance relates to the optimal allocation of capital assets to portfolios. In recent times this field has received renewed attention as traditional portfolio optimisation methods were found to inadequately capture the nongaussian and interdependent nature of the returns of capital assets. A particular case is that of infrastructure assets, which exhibits particularly nongaussian and interdependent returns.

In this thesis we introduce a portfolio choice method developed for nongaussian and interdependent assets and for longer investment horizons, as is common to infrastructure investment. Starting from the classical financial economic assumption of an expected utility maximizing investor, we derive an analytical solution, which incorporates all higher moments of the assets' distributions without making limiting assumptions to ensure solvability. Rather than imposing subjective probability beliefs to infer the return's distributions, we employ Independent Component Analysis to perform a decomposition of the asset space. In this way we are able to identify the fundamental drivers of the returns data and base our portfolio selection on their nature and interdependence.

We apply the method on two samples of infrastructure assets. Firstly, we consider global infrastructure indexes. Secondly, we consider a large sample of airport operators, an asset class of particular interest to this thesis. In both cases we show how the method will outperform its principal rival and contestant, the standard mean-variance optimised portfolio.

The thesis concludes by showing how the method also allows for a redefinition of the concept of diversification, fully integrated with the portfolio choice method. The thesis therefore contributes to the current state of the art and might lead to further research and discussion regarding the possible use of techniques like Independent Component Analysis to solve longstanding questions in theoretical and applied finance.

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Glossary

$R = (R_1, \ldots, R_n)^T$: *Return vector of asset returns*

σ_{ij}: *Covariance*

$\alpha = (\alpha_1, \ldots, \alpha_n)^T$: *Portfolio weights*

r_p: *Portfolio returns*

W: *Wealth*

θ: *Risk aversion coefficient*

π: *Weight of risk-free asset*

P: *Portfolio*

$F(.)$: *Joint cumulative distribution function*

$f(.)$: *Probability density function*

$U(.)$: *Utility function*

$E[.]$: *Expectation*

$V[.]$: *Variance*

μ: *Vector of means of the various portfolios*

β: *Financial elasticity*

α: *Active return on an investment.*

Y: *Vector of state variables*

$S_1, \ldots, S_n$: *Independent components*

a_{ij}: *Mixing coefficients of the independent components*

$kurt(.)$: *Kurtosis*

$Skew(.)$: *Skewness*

$sech$: *Hyperbolic secant*

exp: *Exponential function*

Log: *Natural logarithm*

ζ^*: *Infinitesimal distance*

n: *Number of considered assets*

N: *Number of considered moments*

$DD(P)$: *Diversification Delta*

H: *Information Entropy*

X: *Random variable*

To

A.M.V., M.S.H.V., A.J.M.V., H.P., A.T.N.V., C.E.M.V. and M.E.M.

Acknowledgements

I would like to express my sincere gratitude to my supervisor Professor Francesca Medda and co-supervisor Dr. Thomas Schröder as well as Professor Ben Heydecker for their tireless support and guidance all throughout this Ph.D. degree.

I would equally like to thank University College London, the department of Civil, Environmental and Geomatic Engineering as well as the EPSRC for providing an academic framework stimulating creative thinking in the interest of the advancement of the science.

My sincerest gratitude goes to my closest family, my mother, grandparents, uncle and brothers, for their motivational support year after year.

Sponsorships and Grants

The work for this Ph.D. thesis was conducted in the context of the EPRSC grant EP/H004505. The objectives of the grant were to present an alternative methodology or "toolkit" for the investment in infrastructure with a particular focus on airports. The grant was shared between three civil engineering departments at three UK universities.

At UCL we undertook to address to financial aspects of the investment toolkit focussing on the sustainability and efficiency of airport investment decisions on the one hand and the optimal construction of airport groups or portfolios, to achieve an optimal structure when long-term infrastructure investment is considered. It is the latter which has become the focus of the present work, as the former was the focus of a separate Ph.D. degree.

Introduction

One of the principal questions in finance relates to the optimal allocation of wealth to financial assets in order to "sustain lifetime consumption and bequest" Detemple (2012). Optimal allocation of wealth forms the primary link between financial theory and microeconomics as it relates decisions of intertemporal consumption to the question of the investment of wealth, often referred to as consumption portfolio choice. The simple allocation of wealth between consumption, saving and investment precludes, however, a series of related questions, one of which, the selection of portfolios, is the key focus of this thesis.

Equilibrium behaviour ensuing from the market wide optimisation of the consumption portfolio problem has been the focus of financial economic literature over the past sixty years. The initial framework used in this context, mean-variance (EV) analysis introduced by Markowitz (1952), has since been criticised for its deterministic single period formulation Merton (1969). EV does not allow either for dynamic hedging components aimed at ensuring against fluctuations in the opportunity set, or the set of possible optimal investments, with regard to the risk appetite of the investor.

The focus of financial economic literature on intertemporal investment and consumption decisions has somewhat relegated the creation of the opportunity set to a secondary role. In recent times, work by Jondeau and Rockinger (2006), Jurczenko and Maillet (2006), Harvey, Liechty, Liechty and Müller (2010) and Garlappi and Skoulakis (2011) among others, who will be discussed later, has nevertheless shed new light on the topic. Central to all of these studies is the aim to derive a portfolio choice model that tackles the main concerns of the EV model, a two-step optimisation model which takes its input parameters for granted and which limits the financial world to a two-dimensional risk-return plane.

In the present thesis we build on this recent stream of literature in the field of optimal portfolios as defined by the maximum expected utility criterion. As such our portfolio choice method is an integral part of classical financial economic theory. Even though recent literature in the field of behavioural economics and prospect theory as pioneered by Kahneman and Tversky (1979) has voiced concerns about the empirical validity of the expected utility criterion, it remains the market benchmark and is therefore of important practical interest to this thesis.

More important, however, is the theoretical importance of the criterion, which leads to the three minimal requirements for the portfolio choice method. Firstly, the model should be a fully analytical solution to the maximum expected utility criterion. Secondly, it should address the main concerns raised by the EV model; these are (1) to avoid parameter uncertainty and (2) ensuring the optimality of the chosen portfolio in the presence of higher order moments. Thirdly, the concept should have empirical validity as proven by some empirical applications. In the particular case of this thesis, it is applications in infrastructure finance and airport investment to which it should be geared.

The portfolio choice method is derived over several chapters of this thesis, which are divided over three parts. The first part of the thesis, entitled "Portfolio Choice", recounts in brief the evolution over the past sixty years of portfolio choice methods and their importance within the financial economic literature. At the centre of this evolution has been the quest for an optimal equilibrium model, derived for the optimisation of saving and consumption.

The different chapters map out its evolution and highlight how portfolio choice methods, be it for the choice between risk free and risky assets, or for the composition of risky portfolios, have mostly relied upon a two-dimensional risk-return description of the assets, thereby attaching less importance to higher order moments of the distributions of the asset returns. These higher order moments represent the non-normal behaviour and interrelations of the considered capital asset. In particular, when considering capital assets, which are outside the usual spectrum of investments, such as infrastructure assets, the literature shows that these higher order moments play a crucial role.

The review of the existing literature therefore leaves us with a clear objective of defining a portfolio choice method that is specifically geared towards the characteristics particular to capital assets, which in our case are infrastructure assets. Such a method is presently only partially described in the literature. Chapter 8 is dedicated to showing how infrastructure assets are in fact nongaussian in their behaviour and thus require portfolio choice methods which exceed the two dimensional risk-return framework. The nongaussianity surpasses the marginal distributions and leads to higher-moment dependencies that warrant adapted higher moment portfolio selection methods.

The second part of the thesis is fully dedicated to the derivation of the portfolio choice model. Starting from the traditional maximisation of expected utility, a series of unsolvable multiple integrals is found. In order to pass this hurdle, a change of variables is thus proposed, but the likely candidate would have been Principal Component

Analysis (PCA), however, given the limitations of its reliance on the correlation matrix, the PCA approach would be situated well within the spectrum of traditional mean-variance methods. Instead we opt for a somewhat lesser known technique which has seen some successful applications in finance but which remains mostly unknown of the financial community.

The technique in question is Independent Component Analysis. Contrary to Principal Component Analysis, it works directly from the data assuming it to be a linear mixture of unknown, but completely independent, components that can be retrieved under the most general assumptions. Contrary to Principal Component Analysis, it is not a dimensionality reduction method, as it keeps the dimensions of the problem equal. Rather, it decomposes the mixture of unknown components cross-sectionally. In this way the elements common to a particular driver of the data will be grouped within a single component.

Given the independence of the components, the expected utility maximisation problem can be factorised. The properties of the independent components subsequently help to clean up the mathematical problem and find portfolio choice models which do not impose any restriction other than the applicability of Independent Component Analysis in the decomposition of financial data.

At this point it is important to understand that the obtained portfolio choice models are single period models, and can solve the expected utility maximisation problem fully analytically. The choice of this framework is intentional due to the type of capital assets for which the models have been designed. These investments are longer-term investments in peculiar assets, of which the return characteristics are uncovered with difficulty. A portfolio choice method based on a cross-sectional decomposition of these returns is therefore a first step in the right direction. Such a method does not directly act upon the stochastic nature of the returns, however, it acts upon the probabilistic nature of the possible outcomes in the context of longer-term investments.

The developed models are twofold and span the complete spectrum of utility functions. Additionally, the concept of diversification, which so neatly fits in the EV framework because it shares its definition with the definition of risk, is also maintained in our models and is redefined using the newly-formulated portfolio choice models.

The last part of the thesis presents a series of applications. Two datasets are selected in order to adequately test the properties of the newly-developed methods, the first is a series of infrastructure indexes, and the second dataset is a large sample of airport operators around the world. In both applications the results are indeed promising when compared with the EV portfolios. Both the in and out-of-sample behaviour of the models show how the proposed concepts could aid the infrastructure investor by better capturing the true nature of the returns and building portfolios that are more robust and can resist to periods of financial turmoil.

The last section of the thesis is a summary and conclusion. The aim of the thesis is to present a fully analytical solution to the expected utility maximisation problem is achieved, but the outcome represents only a first step. Many improvements and paths for future research can be envisioned. The method also requires further testing and wider applications such as for example. Generalising the methodology to an intertemporal framework, or extending it to non-linear independent component models, in order to generalise the Independent Component Analysis literature are a few examples. This thesis introduces an alternative concept and research direction, which as was shown, provides interesting and promising results in the form of portfolio choice models in addition to several possible extensions and further applications of the techniques used.

Overview of the structure of the thesis

Part 1: Portfolio Choice
Three chapters detailing the main evolution of portfolio choice and the principal gaps to be addressed
Part 2: Portfolio Choice with Independent Components
Over the course of four chapters we derive two portfolio choice models based on ICA and demonstrate how it leads to a better understanding of infrastructure portfolios
Part 3: Applications in Infrastructure Investment
In the final section we test the model and establish its empirical validity when infrastructure portfolios are considered

Any intelligent fool can make things bigger and more complex... It takes a touch of genius - and a lot of courage to move in the opposite direction.

Albert Einstein

Part 1

Portfolio Choice

1

Current Advances and Challenges in Portfolio Choice

1.1 Introduction

One of the principal questions in finance relates to the optimal allocation of wealth to financial assets in order to “sustain lifetime consumption and bequest” Detemple (2012). Over the past six decades several methods have been developed and we provide here a brief overview. Mean-variance (EV) analysis as presented by Markowitz (1952) has long been the most popular approach to portfolio choice, but was later challenged for its negligence of dynamic hedging components aimed at ensuring against fluctuations in the opportunity set.

The shortcomings of EV, highlighted by Merton (1969) and Merton (1971), point to the deterministic nature of the opportunity set as input parameters are not stochastic and independent of the investor’s horizon. Instead, Merton proposed a partial differential equation (PDE) characterisation of the value function associated with the portfolio characterisation. Since high-dimensional problems lead to unsolvable PDEs, the introduction of a probabilistic approach, using martingales would lead to the identification of optimal consumption portfolios Pliska (1986), Karatzas, Lehoczky and Shreve (1987), Cox and Huang (1989). Stochastic differential equations (SDE) have subsequently been used by Ocone and Karatzas (1991), Detemple, Garcia and Rindisbacher (2003) using Monte Carlo simulations for their implementation.

The aforementioned paragraphs trace three principal evolutions in portfolio choice, which form the content of this section and the next chapter. In parallel to the shift from deterministic to stochastic models, several authors have contributed to a fourth movement characterised by queries into the accuracy of the decision criterion, the expected utility, and the accuracy of its representation through the mean and the variance, whether they are stochastic or not. It is in this setting that the contribution of the thesis should be inscribed.

A brief but detailed review will next be given of the EV model and its founding rational. The shortcomings and the opportunities of the EV model for further innovation will be set out and clarified.

1.2 Mean-Variance analysis in brief

Three fundamental insights form the basis for the EV model introduced by Markowitz (1952) and Markowitz (1959). Firstly: the benefits of diversification as understood by Bernoulli in his 1738 paper on the St. Petersburg paradox, Bernoulli (1954); secondly: its measurement through variance of an asset's realisations, as introduced by Ficher (1906) and later by Marschak (1938); and thirdly: Leonard J. Savage's description of the rational agent's decision making following probability beliefs. The starting point for the train of thought in this case was the explicit link between stock price and the future stream of dividends, as presented by John Burr Williams.

Markowitz (1952) follows John Burr Williams' explicit link between stock price and future stream of dividends, by introducing a normative two-step approach to portfolio choice. An investor first defines his or her probability beliefs regarding all possible capital assets, which are inherently subjective, but combine as objective probabilities would. Markowitz considers this step completed and is primarily concerned with the second step, the definition of the set of investment opportunities. This set consists of all possible combinations of any of the capital assets in varying proportions in a portfolio. These combinations that maximise the expected return for a certain level of expected risk, or variance, from an "efficient set". The theory to which this approach led became known as Modern Portfolio Theory or MPT.

The asset space from which the investor will choose a portfolio consists of n risky assets $R_i, (i = 1..n)$ with return vector $R = (R_1, \dots, R_n)^T$ and joint cumulative distribution $F(R_1, \dots, R_n)$. Let the vector $\alpha = (\alpha_1, \dots, \alpha_n)^T$ represent the fractions invested in each of the various risky assets. Let $E[R_i]$ be the expected return of an asset R_i, $V[R_i]$ is the variance of asset R_i and r_p is the return of the portfolio. From the properties of the expectation function we know that the expected return of a portfolio can be given as follows:

$$E[r_p] = \sum_{i=1}^{n} \alpha_i \; E[R_i] \tag{1.1}$$

Similarly, the variance of the portfolios can be given by the following relation:

$$V[r_p] = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \, \alpha_j \, \sigma_{ij} \tag{1.2}$$

where σ_{ij} represents the covariance of two assets i, j and σ_{ii} represents the standard deviation of asset i. In Merton (1972) analytically derived the shape of the efficient set to be hyperbolic, enclosing inside it all possible inefficient portfolios built with the assets present in the investment universe.

Using matrix notation, we extend to the consumption portfolio invested in an efficient portfolio P with return r_p and a risk free asset in proportion π, and obtain the following relation for an initial wealth W_0 and a final wealth W:

$$E[U(W)] = E[W] - \frac{1}{2\theta} V[W] \tag{1.3}$$

Subject to wealth constraint:

$$W = W_0 \pi^T (r_p - r1) + W_0(1 + r) \tag{1.4}$$

where θ is a coefficient of investor risk aversion and r is the return generated by the risk free asset. Maximising the relation, permitting short-selling leads to the following first order condition:

$$E\left[(r_p - r1) - \frac{1}{\theta}(r_p - r1)(r_p - r1) W_0 \pi\right] = 0 \tag{1.5}$$

which solves to:

$$W_0 \pi = \theta V^{-1}(\mu - r1) \tag{1.6}$$

μ is the vector of means of the various portfolios P on the efficient set eligible for selection in the portfolio. Relation (1.6) thus determines the fraction of initial wealth to be invested in the portfolio and the riskless asset.

The question of the choice of portfolio p was solved by Sharpe (1964) who showed that the line linking the riskless asset with the optimal portfolio will have the greatest tangent. It led to the introduction of the equilibrium model in order to price capital assets, the CAPM, in which the tangent would be referred to as the security market line. The portfolio maximizing the tangent is the market portfolio that generates the market rate of return.

Contrary to Tobin (1958) who introduced the maximisation of expected utility as a decision criterion for optimal portfolios, EV is not explicitly derived as an expected utility maximising model. The EV model is only formally linked to expected utility maximisation in the work of Levy and Markowitz (1979). EV rests upon a simple two-dimensional framework consisting of the first two moments of the probability density function of the assets' returns distribution. Probability theory provides the tools to estimate those moments adequately, given the probability beliefs of the investor. The interdependence between assets is thought to be a linear one, expressed by Pearson's correlation coefficient. Additionally, the substitution rate of expected return for held risk is a marginally decreasing one, implying increased risk adjusted returns.

Roll (1977) was among many to highlight the weaknesses and challenges faced by EV, which will be the focus of the next section. Nevertheless, the modern theory put finance on the map as a separate, though not yet completely defined, field of economics, making the theory one of the most significant contributions to the science. It is its simplicity and clarity, which has allowed it to remain widely applied in the practical world, despite its shortcomings.

1.3 Multi-period EV models

Markowitz (1952) clearly states in his original paper that the probability distributions of the considered assets are likely to change over time. As a consequence the subjective probability beliefs that form the basis of the asset allocations will not remain constant either. Several authors have therefore generalised the portfolio choice framework to a multi-period setting, (see Mossin (1968), Samuelson (1969) Hakansson (1971), Merton and Samuelson (1974), Campbell, Lo, and MacKinlay (1997), Steinbach (2001), Leippold, Trojani, and Vanini (2004), Brandt and Santa-Clara (2006), Basak and Chabakauri (2010)).

Discrete multi-period solutions to portfolio choice have proved to be much more elusive than continuous extensions, to which we will turn our attention later. Until recently, no closed form solution to the discrete multi-period model has been provided in literature, unless independence is assumed, as is the case for Li and Ng (2000) and Leippold, Trojani and Vanini (2004). Bodnar, Parolya and Schmid (2012) do provide an analytical solution to the discrete problem, but not without imposing fairly important restrictions.

Bodnar, Parolya and Schmid (2012) limit the assumptions to the existence of a conditional mean vector and a conditional covariance matrix and thereby avoid the hypothesis of independence, when deriving a solution to the asset allocation problem with and without a riskless asset. Yet their model is derived for the quadratic utility only. Even if Brandt (2006) considers the quadratic utility to be a good approximation of the utility curve, it remains an approximation and part of only one type of utility curves. More general solutions to the problem do exist, but contrary to those defined above, they are not analytical (see, van Binsbergen and Brandt (2007)).

1.4 The shortcomings of the EV model

EV models owe their popularity to their intuitiveness and simplicity as well as to a new way of thinking about financial portfolios Rubinstein (2002). EV models as discussed above were derived as a normative theory in which asset allocation decisions are made following deterministic estimates of the mean, the variance and the covariance, presumed being given ex-ante. The structure of the theory prescribes investor preferences to be accurately captured by the mean and variance of the assets, or a Taylor series approximation of the utility curve using the first two terms, the first two moments, as illustrated by Levy and Markowitz (1979).

Three main drawbacks should be highlighted at this stage. Firstly, EV models inadequately describe financial data. In light of findings and stylized facts relating to financial data by Kendall and Bradford Hill (1953), Mandelbrot (1963), Fama (1963), Blattberg and Gonedes (1974), Kon (1984), Loretan and Phillips (1944), Longin (1996) and Cont (2001), it has become apparent that financial data require a more accurate description of risk that reaches beyond variance, or beyond second-order moments.

Contributions in this field have been made by Jean (1971) and Schweser (1978) who proposed methods that would include higher moments in the description of risk. Several other authors extended the work presented above, Homaifar and Graddy (1988) and Fang and Lai (1997), by an extended mean-variance framework including skewness. Diacogiannis (1994) and Athayde and Flôres (1997) have greatly simplified the moment expressions, which thereafter helped to solve the optimisation problem numerically, as higher moments would lead to analytically unsolvable problems. Another example is presented by Lai (1991) and Chunhachinda, Dandapani, Hamid, and Prakash (1997), who introduced Polynomial Goal Programming (PGP) in order to solve mean-variance-skewness based optimisation problems.

The second important drawback relates to the definition of investor preferences. Markowitz (1952) derived the EV model using the expected return, approximated by the mean and the variance as a proxy for investor preference. Only at a later stage was this related to classical financial economics and the maximization of the investor's expected utility (MEUC) in Levy and Markowitz (1979). They show several commonly used von Neumann-Morgenstern (vNM) utility functions, which, when approximated by a two term Taylor series, lead to EV optimal portfolios. The problem, however, is that this approach leads to negative marginal utility of wealth, which means that additional wealth becomes undesirable as noted by Meucci (2005).

This apparent disconnect from classical financial economics was intentional. Mathematically, expected investor utility leads to analytically unsolvable multiple integrals. We assume that an investor holds an initial wealth of W_0, arbitrarily fixed to 1 at the beginning of the considered period. The end-of-period wealth is denoted, W, from interval $I \subset \mathbb{R}\ to\ \mathbb{R}$ and a von Neumann-Morgenstern utility function $U(.)$ is defined over W, which defines the investors preferences. The asset space from which the investor will choose its optimal portfolio, consists of n risky infrastructure assets with return vector $R = (R_1, \dots, R_n)^T$ and joint cumulative distribution $F(R_1, \dots, R_n)$. End-of-period wealth can be represented by $W = 1 + r_p$, with $r_p = \alpha^T R$, where the vector $\alpha = (\alpha_1, \dots, \alpha_n)^T$ represents the fractions of wealth invested in each of the various risky assets.

$$\max_{\alpha_i} E[U(W)] = \int U(W)f(W)dW = \int \dots \int U(1 + \sum_{i=1}^{n} \alpha_i R_i)dF(R_1, \dots, R_n) \tag{1.7}$$

These multiple integrals usually lead to problems, which do not have closed form solutions. Additionally, there is no bijective relation between expected utility theory on the one hand and higher moments of distribution functions on the other hand, thereby again introducing a theoretical divide between EV and its higher moment extensions and the use of a decision making criterion grounded in financial economic theory.

In order to guarantee conversion of relation (1.7), three conditions must be fulfilled regarding both the utility function and the probability density function of the returns, as explained by Loistl (1976), Lhabitant (1997) and Jurczenko and Maillet (2006). We will delve into the mathematical details of these conditions in a later chapter. The conditions concern the translation of investor preferences into the first N moments of the investment return distribution.

Firstly, the utility function should be an analytical function of returns at the point of the expected returns, while realised returns must remain inside a convergence interval in order for a relation to exist between the series of moments and the moment of the expected returns. See, Tsiang (1972), Loistl (1976) and Lhabitant (1997). Secondly, the convergence interval should be shrunk slightly for a Taylor series approximation of the utility function to converge uniformly, and for the integral and summation operators to be interchangeable. See, Loistl (1976), Lhabitant (1997) and Christensen and Christensen (2004). Thirdly, the Hamburger (1920) moment condition must be respected, with all even moments to having a negative sign and all uneven moments being positive.

Based on these conditions several expected utility maximising models for asset allocation have been proposed. Prominent amongst those are Levy (1969), Hanoch and Levy (1970), Rubinstein (1973) and Kraus and Litzenberger (1976) and more recently, Jondeau and Rockinger (2006) and Harvey, Liechty, Liechty, and Müller (2010). Common to all of these models is the use of a Taylor series expansion to approximate the utility function or the use of a polynomial utility function which achieves essentially a similar result. Equal in all the above cases is the truncation of the Taylor series approximation to those moments, which the authors deem necessary to accurately capture investor preferences and which can reasonably be regarded as part of such preferences.

All the afore-mentioned models examined the theoretical principles behind EV in relation to classical financial economics without changing the main premises: a single period normative model, which assumes that rational agents optimise a decision criterion. This leads to the third and final drawback of the EV model. Markowitz himself eluded to the fact that probability density functions change over time thereby highlighting the weaknesses of a deterministic model. Markowitz's criticism in fact leads to three streams of literature. First is a stream spearheading the move away from direct or approximated expected utility maximisation through the introduction of Stochastic Dominance models (SD) and Dynamic Portfolio Choice models (DPC). Both of these models will be discussed in the next chapter.

The second and third streams of literature focus on the determination of the input parameters for the EV models. On the one hand, the deterministic nature of the input parameters presents an issue, which will be discussed below. On the other hand, a vast literature has developed around what has become known as parameter uncertainty, or put differently, the uncertainty related to the estimation of input parameters or determination of the probability beliefs of the considered assets. That literature goes beyond the scope of this chapter. However, one specific element of it, the estimation of the covariance and correlation structure, is both characteristic of EV and is an essential aspect of the remainder of this work.

With EV, financial economics was confronted with the necessity to efficiently estimate covariance and correlation structures on large scales. Sharpe (1967) formally introduced the principle of using index models to achieve this effect. These models would relate the returns of the considered asset to the returns of the "market", as represented by an index. The method gave rise to the concepts of the asset's alfa (α) and beta (β) coefficients relating the asset's return R_{it} to the market's return R_{mt} as shown in equation (1.8). The residuals are presumed to be uncorrelated, an assumption that was contradicted by Fama (1968). Index models had immediate success because asset managers found it much easier to relate the evolution of an asset to that of the market rather than to the evolution of a different asset from a totally different asset class. Another main advantage was that the index models reduced the number of estimations necessary from $\frac{n(n-1)}{2}$, to $(n+1)$, where $n, with\, n > 3$ represents the number of considered assets.

$$R_{it} = \alpha_i + \beta_i R_{mt} + \epsilon_{it} \tag{1.8}$$

In their book, Elton and Gruber (1973) show how the model could more accurately estimate the covariance using historical data rather than direct estimation. Single-index models subsequently got extended to include multiple indexes that represented not one but several aspects of the market. The question arose as to how many indexes there are and how they could be extracted. Both ex-ante and ex-post methods were developed in which factors or indexes were extracted from the variance-covariance matrix. Principal Component Analysis and Factor Analysis have been shown to be key techniques in this area (see, Roll and Ross, (1972), Cho, Elton and Gruber (1984), Dhrymes, Friend and Gultekin (1984) and Brown and Weinstein (1978)).

Three types of pre-specification models in particular have been developed: market plus industry models as defined by Cohen and Pogue (1967), those which account for surprise in basic economic indicators such as production or inflation by Chen, Roll, and Ross (1986); and portfolios of traded securities such as large minus small, defined by Fama and French (1992). The use of pre-specification models gained acceptance due to their predictive capacity in accurately capturing the true nature of the covariance structure as shown for instance by Elton and Gruber (1973).

Ross (1978) and Ingersoll (1987) have reformulated EV for multi-index models. This effectively transformed the optimisation problem by adding one axis per beta associated to each of the indexes. In principle this approach still corresponds to EV, even though it might present a more practical framework for investors to understand the dependencies of the assets on state variables. In addition, an axis would be added in case of mispricing of the assets and the risk related to it. This brings us to the last important section of this analysis of EV, the relation between theory and application.

Beyond its theoretical aspects, the practical applicability of EV became the next focal point for criticism. The computational efficiency of large-scale optimisation problems presented a challenge to the computational power of the day. Therefore Elton, Gruber and Padberg (1976), Elton, Gruber, and Padberg (1977), (Elton, Gruber and Padberg (1978) provide ranking criteria for the inclusion of assets in the composition of portfolios, even though these ranking criteria no longer correspond any longer to current portfolio models.

Given the normative and deterministic nature of the EV model and the limitations of the available estimation techniques, the literature has proposed several ways to deal with estimation inaccuracies. The literature on these topics can roughly be split into two parts, the practical sensitivity to input data, secondly, how portfolios in practice deal with those sensitivities. The sensitivity of EV to its input parameters is well documented. See Jobson and Korkie (1981), Michaud (1989), Best and Grauer (1991), Broadie (1993), Chopra and Ziemba (1993) and which is perhaps best summarised by Chopra (1993) who has shown that the extreme sensitivity to even the slightest changes of estimated return and risk leads to large variations in the optimal portfolios.

In dealing with these challenges in a practical manner, several solutions were found for the estimation of input variables to create mean-variance stable portfolios. James-Stein shrinkage estimators were introduced by Jobson and Korkie (1981), Jorion (1985) and others, shrinking the expected return estimator towards the minimum variance portfolio. Black and Litterman (1990) chose a Bayesian approach to produce stable expected return estimates. Michaud (1998) on the other hand introduces a resampling method to achieve the same result even though this method is somewhat ad hoc and falls prey to many pitfalls as explained by Scherer (2002).

In more recent times, robust statistics Cavadini, Sbuelz, and Trojani (2002) and robust portfolio optimisation have been introduced Pachamanova (2006) and Fabozzi, Kolm, Pachamanova and Focardi (2007). These techniques consider the estimation errors directly within the optimisation itself. The technique finds its origin in Ben-Tal and Nemirovski (1997) for robust truss topology design, and was subsequently introduced in portfolio finance.

The common denominator in all of the above-mentioned techniques is the quest for more reliable optimal portfolios. None of the techniques fundamentally changes or questions the method for optimising the portfolio itself. In that respect they fit in the third and remaining window of criticism and solutions to EV, which considers the practical applicability of the theory.

1.5 Recent advances in portfolio choice with higher moments

Since the introduction of the EV models by Markowitz (1952), financial theory has moved from a deterministic normative setting to a stochastic description of the consumption portfolio following the complete market model, see Detemple (2012). As accuracy increased so also did complexity, in the quest for a model to describe the equilibrium behaviour of rational utility maximising agents, and to thereby unify financial economic theory with a description of optimal asset allocating behaviour. Dynamic portfolio choice models have come close to achieving this result.

In recent years special attention has been placed on the inclusion of higher moments in portfolio choice models as well as on decision criteria other than MEUC. Such focus can be explained on the one hand by the fact that even stochastic models, when based only on the mean and the variance, are not always sufficient to capture the true nature of the asset returns. On the other hand, an increasing body of literature contesting the validity of the MEUC explains the tentative shift away from classical financial economics. Appendix 1 provides an overview of this work.

In that context a significant body of work is developing, focussing in the first instance on the selection of the portfolios of risky assets. Questions around the equilibrium behaviour of the economic agents, market portfolios and fund separation theorems are either considered to be solved by such papers as Cass and Stiglitz (1970) or as a second step first requiring an accurate portfolio choice model for risky assets. A good example of such an approach is Hwang and Satchell (1999). The first objective of this literature is therefore to find portfolio choice models that increase the accuracy with which the specificities of asset returns are captured.

Central to the renewed interest in portfolio choice is the inclusion of higher moments of the assets' return distributions. The inclusion of higher moments implies the extension of the definition of risk, beyond the variance, as required by a growing body of empirical evidence regarding financial returns, see Cont (2001). The inclusion of higher moments equally leads to an increased accuracy in the definition of investor preferences.

The extension to higher moments is, however, not trivial. The MEUC leads to a series of unsolvable multiple integrals. Following Levy and Markowitz (1979), the primary method applied has been the approximation of the utility curve by a function of the moments. To maintain solvability, this function, usually a Taylor series, was truncated after the second term, the variance, leading to a series of 2 moments. In recent literature this approach has been revisited and extended. We will consider three papers in particular to illustrate these advances; first of all Jondeau and Rockinger (2006) secondly Jurczenko and Maillet (2006) and lastly Garlappi and Skoulakis (2011).

Jondeau and Rockinger (2006) present a classic case of MEUC portfolio choice with higher moments and take as their starting point a classical definition of the MEUC:

$$E[U(W)] = \int U(W)\, f(W) dW \tag{1.9}$$

where $f(W)$ is the probability density function of the end-of-period wealth of the portfolio and $U(W)$is the utility function. Approximating the utility curve by a Taylor series around $E[W] = \bar{W}$ and truncating the series at the fourth term while discarding the error term:

$$U(W) = \sum_{k=0}^{4} \frac{U^{(k)}(\bar{W})(W - \bar{W})^k}{k!} \tag{1.10}$$

the complex multiple integral in equation (1.9) can be simplified to a discrete optimisation problem:

$$E[U(W)] = \sum_{k=0}^{4} \frac{U^{(k)}(\bar{W})}{k!}\, E[(W - \bar{W})^k] \tag{1.11}$$

Jondeau and Rockinger (2006) choose an exponential utility function and derive the optimal composition of an asset portfolio of risky assets, showing that the inclusion of higher moments reduces the opportunity cost of approximating the utility curve as compared to direct optimisation. As such, they prove that higher moments add to the definition of risk and the accuracy of the approximation of both the utility and the return distributions.

Yet their definition and approach are incomplete. Jondeau and Rockinger (2006) limit their model to the exponential utility function and fail to formally justify convergence of their approximation of utility with a truncation of the series after the fourth term. Such definitions follow in the work by Jurczenko and Maillet (2006) on higher moment asset allocation and pricing models. Several other models assume the investor's expected utility is well approximated by inserting estimates of the moments of an assumed sampling model[1].

Three conditions should be met when considering the approximation of equation (1.9) by a Taylor series of the form expressed in equation (1.10). Firstly, the utility function chosen should be an analytical function at the point of interest, $E[U(W)]$, while the realised returns must remain within the convergence interval of the infinite order Taylor series of the utility function as shown by Tsiang (1972), Loistl (1976) and Lhabitant (1997). Secondly the convergence of $U(.)$ around $E[R]$ towards $U(.)$ should be a uniform one. This is a condition introduced by Loistl (1976), Lhabitant (1997) and Christensen and Christensen (2004) and allows for the interchangeability of the integral and summand operators in the objective function of the optimisation. Thirdly, the

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[1] The importance of higher moments has been investigated in detail by authors including Campbell et al. (2001), Chen et al. (2001), Dittmar (2002), Athayde and Flores (2004), Burger and Warnock (2004), Goetzman and Kumar (2004), Levy and Levy (2004), Patton (2004), Adcock (2005), Brunnermeier and Parker (2005), Liew and French (2005), Sfiridis (2005), Ang et al. (2006), Bakshi and Madan (2006), Barro (2006), Williams and Ioannidis (2006), Barberis and Huang (2007), Briec et al. (2007), Brunnermeier et al. (2007), Chiang and Li (2007), Guidolin and Timmermann (2007), Mitton and Vorkink (2007), Martellini and Ziemann (2007), Chabi-Yo (2008a, b), Cvitanic´ et al. (2008), DeMiguel et al. (2009, 2010), Post et al. (2008), Bacmann and Benedetti (2009), Da Silva et al. (2009), Hall et al. (2009), Knight and Satchell (2009), Mencia and Sentana (2009), Morton and Popova (2009), Wilcox and Fabozzi (2009), Zhou (2009), Bali and Cakici (2010), Blau and Pinegar (2009), Brandt et al. (2010), Conrad et al. (2010), Fabozzi et al. (2010), Martin (2010), Poti (2010), Vorkink et al. (2009).

Hamburger moment condition, Hamburger (1920), should be respected; the implication here is that even moments should be minimised while uneven ones should be maximised. In chapter 6 of the present work we will look into the mathematical detail of these conditions.

These conditions allow for the formal definition of a higher four-moment asset allocation model based on a Taylor series approximation of the utility function. In this way both the definition of risk is enlarged and the description of investor preferences is broadened. However, portfolios based on this utility maximising framework now depend not only on higher moments but also on higher order co-moments, thus adding to the complexity of the estimation and parameter uncertainty. Gains in accuracy are potentially undone by the added complexity of a deterministic framework, which is now requiring many more parameters per asset.

In light of these considerations Harvey, Liechty, Liechty, and Müller (2010) consider a Bayesian approach to portfolio selection. A Bayesian probability model is used for the joint distributions of asset returns. This Bayesian framework is subsequently used to maximise the expected utilities using predicted returns. In this way the parameter uncertainty is avoided while higher moments are included in the maximisation of expected utilities (see Adcock (2005), Ang, Chen, and Xing (2006), Athayde & Flores (2004), Bacmann and Benedetti (2009), Bakshi and Madan (2006), Bali and Cakici (2010), Barberis and Huang (2007), Barro (2006), Blau and Pinegar (2009), Brandt, Brav, Graham, and Kumar (2010), Brunnermeier and Parker (2005), Brunnermeier, (2007), Burger and Warnock (2004), Campbell, Lettau, Malkiel, and Xu (2001)).

Even if the use of a Bayesian framework addresses part of the problems highlighted for the maximisation of Taylor series approximations, it nevertheless requires the ex-ante selection of a distribution function from which the predicted returns are deduced. In this sense the method cuts corners of the inclusion of all characteristics of financial data, which, according to Cont (2001) go beyond the skew-normal distribution.

The last paper to be considered is Garlappi and Skoulakis (2011) as it introduces the use of a tranformation of the data to increase the chance of convergence of the Taylor approximation of the utility function. The paper introduces the idea of usng a nonlinear transformation so that the base of the distributions of the considered assets is reduced and the distributions them selves are rendered symmetrical. This idea is a powerful one and will be considered in detail in later chapters.

The three papers discussed above have addressed the higher moment portfolio problem by starting from a standard deterministic MEUC in a single period framework and extending it to include characteristics of empirical financial data. In doing so they address the problem only partially and either increase the concerns related to parameter uncertainty or by limiting the effective inclusion of higher moments through a parametric choice of predictive distributions in a Bayesian context. Therefore, a true solution to the expected utility maximising framework, which does not impose additional constraints or introduces further approximations, has yet to be formulated using these three papers as a base.

1.6 Beyond classical financial economics

Beyond classical financial economics, and utility maximising models, several advances have also been made which are worth highlighting at present. Firstly, the financial crisis has had the effect of questioning many of the established frameworks in finance. Secondly, the open door for methods previously foreign to finance and economics and usually introduced from other scientific disciplines, was pulled ever more slightly open. Thirdly, behavioural economics and prospect theory, which review the validity of the utility framework, have gained increased attention.

The main objective in the re-examination of existing theories has been the search for adaptations that will make them more robust to future crises. Some examples would be the definition of a multivariate intertemporal portfolio choice model with a stochastic correlation presented by Porchia and Trojani (2010). The illiquidity of markets is a second important consequence of the financial crisis, thus leading to portfolio models, which are geared specifically towards this illiquidity such as Gârleanu (2009). Asymmetry of information and portfolio choice models adapted to it is a further focal point as presented by Biais, Bossaerts, and Spatt (2010).

A related topic is the question of diversification and its importance, relevance and impact during the crisis. Several recent publications have been dedicated to its re-examination and lead to various important conclusions. As virtually all long only asset classes moved down together during the recent financial crisis, diversification was presumed dead. Ilmanen and Kizer (2012) argue against this conclusion and demonstrate there is ample room for improvement when the focus shifts away from asset diversification and onto factor diversification. Hjalmarsson (2011) arrived at a similar conclusion with the introduction of characteristic based diversification.

The third important consequence of the crisis has been the slow but steady acceptance of new techniques in mainstream financial journals. Examples of this would be Vermorken, Szafarz, & Pirotte (2010), who introduce Independent Component Analysis in order to understand the fundamental relationships between stocks and who analyse how these relations compare to their sector classifications. The use of physics and statistical mechanics is not new in finance; the Black-Scholes model is of course heavily dependent on it. Yet, concepts such as information entropy and techniques such as Independent Component Analysis, have been unknown within the field of finance for long periods of time.

Other fairly recent examples are Frittelli (2000), Rouge and Karoui (2001), Hulabek and Sgarra (2006) and Mistrulli (2011), who have applied information entropy in finance and financial risk management. Another recent contribution will be discussed in a later chapter and forms a new entropy based diversification measure. The increasing popularity of methods from outside traditional finance and statistics indicates a shift in the acceptance of such techniques as a sign of future progress in the field.

Lastly, the utility paradigm has also been targeted repeatedly, especially the introduction of prospect theory in portfolio choice models presented by He and Zhou (2010). Prospect theory, introduced by Kahneman and Tversky (1979), argues that the MEUC has significant flaws, which make it unsuitable as a financial decision criterion. The appendix of this thesis provides an overview of this argumentation. As an alternative, Prospect

Theory provides a kinked utility function with a reference point from which all investors operate. In addition, investors are presumed to think in returns rather than terminal wealth in their allocation of wealth.

Prospect theory has received significant attention and a large following. The empirical literature documenting the empirical validity of the theory when tested in and outside laboratory conditions remains divided, however. This leads to the notion that most financial research in portfolio choice is still conducted in the MEUC framework.

On the basis of our discussion here it is clear that financial economic theory has not yet pinned down a series of frameworks that deal with all the different drawbacks highlighted for the different models presented in the previous chapters. The absence of such a framework presents an opportunity for the introduction of models tailored either to specific assets, or, when the model is deemed sufficiently general, tailored to wider applications in financial economics.

1.7 The gaps to address

When taking stock of the previous chapters the evolution of portfolio choice can briefly be summarised as follows. Markowitz introduced first and foremost a method to select individual assets in the construction of a portfolio and was motivated by the realisation that diversification reduces the risk of an investment. Sharpe (1964) subsequently highlighted how this logic could be extended to an equilibrium model and by describing the behaviour of all market participants in the search for an optimal consumption-portfolio. This portfolio would consist of a risk-free and a risky portfolio, depending on the risk aversion, saving and consumption behaviour of the agent. Realising how this model was based on several limiting assumptions, continuous-time finance extended the theory to a stochastic formulation, capturing, in a stochastic mean-variance framework the consumption, investment and saving behaviour of economic agents.

At this juncture we can identify a series of gaps, which presently require attention from the academic community. First of all, further work is required on the subject of the consumption portfolio and the market equilibrium. This would entail a complete overhaul of financial theory. But before doing so, however, attention should be paid to the construction of the market portfolio itself. Presently, this market portfolio and all portfolios of risky assets principally rely on methods dependent on the EV framework. Several additions to the literature have introduced higher moment portfolio choice methods, but none of which truly captures the nature of the asset's returns without equally introducing drawbacks and approximations in equal measure.

A gap therefore exists in the current state of the art on portfolio choice methods, which can capture all characteristics of a risky asset in the investment universe, while avoiding the introduction of limiting approximations which compromise any efforts in increasing accuracy. In particular when highly specific assets or portfolios are concerned, portfolios for which high frequency data might not be available, or which require longer term investment, a portfolio choice method capable of capturing the true characteristics of the considered assets, selecting in this way a portfolio which is more hedged to shocks to state-variables, could have a significant added value.

When considering the specific asset class on which this thesis is focussed, infrastructure assets, in particular airports and large formerly owned public infrastructure assets, it is clear that so far no dedicated portfolio selection method exists which will capture the true nature of infrastructure returns while taking into account, both the nature of infrastructure investment vehicles as well as the objectives set for such investment vehicles. The next two sections of this thesis will focus on developing a series of methods, which will meet the expectations of a longer-term higher moment portfolio choice model for infrastructure assets.

We can only see a short distance ahead,
but we can see plenty there that needs to be done.

Alan Turing

Part 2

Portfolio Choice with Independent Components[2]

[2] Part 2 is based on Vermorken, Szafarz, and Pirotte (2010), Vermorken, Medda, and Schröder (2012) and a conference paper entitled "Higher moment asset allocation" presented at the 5th annual CSDA conference on computational financial econometrics, held in London on 17-19 December 2011.

2
Independent Component Analysis

2.1 Introduction

The aim of this part of the thesis is to derive over the length of three chapters an alternative portfolio choice model. The purpose of the model is threefold. First of all the model should provide a fully analytical solution to a portfolio choice model for an expected utility maximising investor without imposing limiting assumptions regarding the number of higher moments to be taken into account, or the type of utility function to be used.

Second, by being a higher moment expected utility maximising model it should provide a better understanding of the interdependencies and co-movement between asset returns. The model should therefore require less rebalancing, as joint distributions of asset returns are increasingly understood. This is of particular interest to sectors that invest longer term without the possibility of rebalancing the portfolio. Such long term examples are infrastructure operators and funds, as well as private equity and real estate funds, which typically hold their assets for longer periods of time without the possibility of rebalancing.

The third purpose of the model is to introduce an alternative technique or set of techniques into financial economics. The financial crisis has uncovered how some traditional techniques in financial economics may not be adequate in capturing the full complexity of the market. Several techniques have recently been proposed to better understand the properties of and dependencies between capital assets. One technique in particular, Independent Component Analysis (ICA), a technique used to perform Blind Source Separation, has increased in popularity, as it gradually became understood.

Several recent applications of Independent Component Analysis can be cited and will be discussed later on. In particular Back and Weigend (1997), Hyvärinen, Karhunen, and Oja (2001) and the present one are important examples. All three papers show through different applications, how the fundamental structure of financial data can be laid bare through the use of ICA. As parts of the model could not have been derived had ICA not been available, we have elected to start this section by first reviewing ICA, before returning to portfolio choice models.

The structure of this second part is therefore as follows. We will first discuss in great detail several of the techniques used in later chapters in order to make the derivations of the model as explicit as possible. Subsequently the derivation of the three models presented in this thesis will be discussed. All applications and empirical tests will be presented in a separate section of this thesis.

2.2 Independent Component Analysis

2.2.1 Introduction

The most important technique applied in the derivation of the portfolio choice models is known as Independent Component Analysis (ICA), which forms part of a larger group of mathematical methods commonly referred to as Blind Source Separation, following its principal feature. The technique is part of the field of statistical signal processing and neural networks and was introduced by Jutten (1987), Jutten and Hérault (1991) and subsequently further developed by Comon (1994), Cichocki, Unbehauen and Rummert

(1994), Bell and Sejnowski (1995), Cardoso and Laheld (1996), Amari, Cichocki, and Yang, (1996), Pearlmutter and Parra (1997), Deco and Obradovic (1996), Oja (1997), Karhunen, Cichocki, Kasprzak and Pajunen (1997) and Girolami and Fyfe (1997).

The technique, as its name indicates, was developed to retrieve the sources from their mixed form, the mixed signal. These sources have the particularity of being both statistically independent and retrieved blindly. The implications of the former will be discussed later, however, the latter implies that the retrieval of the sources must be done using only the observed mixed signal.

To achieve blind separation of the sources several theories have been developed. Independent Component Analysis is the most prominent and the one which has received the most attention in recent years, especially due to its wide application in fields ranging from neural computation, medical imaging, telecommunication and others. The process used for retrieving the independent components from the mixed signal is not unique. Several independent component estimation methods have been proposed, and these are presented below. Before doing so, however, understanding the intuition behind ICA is of crucial importance.

Independent Component Analysis defines a generative model. It describes how a set of n random variables $R_1, \dots, R_n$, which represent n assets, are generated by an equal or lower number of independent components $\mathrm{S}_1, \dots, \mathrm{S}_\mathrm{n}$. The relation between assets and components is a linear one and can be described as follows:

$$R_i = \sum_{j=1}^{n} a_{ij} S_j \text{, where } a_{ij} \text{ are real coefficients, } i, j = 1, \dots, n \tag{2.1}$$

The a_{ij} are the mixing coefficients and the elements of a matrix A, the mixing matrix, which describes the relation between asset and components. The independent components are statistically mutually independent by definition. They cannot be observed directly and are therefore latent variables. To guarantee maximal independence of the components, the estimation must be evaluated under the most general assumptions possible. This fact implies that several methods can be used for the estimation and all of them share the one characteristic of maximising nongaussianity.

To understand the intuition behind the ICA application we can turn to the Central Limit Theorem, which states that any combination of nongaussian independent random variables will always tend towards gaussianity under certain fairly general conditions. Conversely for ICA, this implies that, on the one hand, the independent components are distributed following nongaussian distributions and on the other hand, the co-moments present in the joint distribution of the random variables are represented by the independent components.

Contrary to most changes of variables which have as primary objective dimensionality reduction, such as for instance Principal Component Analysis, ICA aims to reduce complexity instead. The decomposition is cross-sectional because it decomposes the joint probability density function associated with the random variables, into independent parts. This step implies that the dependence structure between the random variables $R_1, \dots, R_n$ is now captured within the lines of the mixing matrix A. Achieving a full decomposition is possible via different techniques that are discussed below.

2.2.2 Independent component estimation

Since its introduction in 1987, several authors have developed methods to efficiently estimate the independent components. Common to these methods is the main objective of maximising a measure of nongaussianity as, following the Central Limit Theorem, independence can be achieved through the maximisation of the nongaussianity of the components. This can perhaps best be explained when we examine the simplest method for estimating the independent components using kurtosis as a measure of nongaussianity as in Hyvarinen et al. (2001).

Let us consider $R_i = AS$ a random variable, comprised by components S_1 and S_2 with mixing matrix A. We now look for one of the independent components as $y = b^T R$ and ewe consider the transformed vector $q = A^T b$, with b an unknown vector. We assume S_1 and S_2 have kurtosis $kurt(S_1)$ and $kurt(S_2)$ is assumed to be different from zero. Additionally it is assumed that the independent components are normalised thus implying $\mathrm{E}[y^2] = 1$. From the definition of kurtosis we know it has the following properties:

$$kurt(S_1 + S_2) = kurt(S_1) + kurt(S_2) \tag{2.2}$$

$$kurt(a_1 S_1) = a_1^4 kurt(S_1) \tag{2.3}$$

Then we find $y = b^T R = b^T AS = a_1 S_1 + a_2 S_2$. Using the additive property of kurtosis, we find:

$$kurt(y) = kurt(q_1 S_1) + kurt(q_2 S_2) = q_1^4 kurt(S_1) + q_2^4 kurt(S_2) \tag{2.4}$$

It is evident that based on this principal, an objective function can be constructed in the following form:

$$|kurt(y)| = |q_1^4 kurt(S_1) + q_2^4 kurt(S_2)| \tag{2.5}$$

Finding the maxima of this objective function then leads us to the identification of the independent components.

The technique described above is a very simple one and not used in practice, as kurtosis is sensitive to outliers in the data. The principle used in the described method is, however, similar for all independent component estimation methods, that is to say, a nongaussianity criterion is identified and the ICA estimation problem is reformulated using this criterion for an objective function to be identified.

Largely speaking, two sets of prominent techniques have thus far been developed. On the one hand are the techniques based on information theory, and on the other hand are techniques based on likelihood functions. We will return to the latter in greater detail later on in this section. The former should retain our attention, given the popularity of the techniques. Among them two techniques of particular importance are the minimisation of Mutual Information by Amari, Cichocki and Yang (1996) and FASTICA by Hyvärinen, Karhunen and Oja (2001).

In order to give an overview of both techniques, it is important that we first define the concept of information entropy H of a random variable y with probability density function $f(y)$:

$$H(y) = -\int_{-\infty}^{\infty} f(y) \log f(y) dy \tag{2.6}$$

H is formulated as the expectation of the natural logarithm of the probability density function. Whenever areas of concentration occur and some outcomes are more likely than others, $f(y)$ increases as a consequence. Such areas of concentration decrease the uncertainty of the possible outcomes of a random draw. The entropy will therefore decrease when a distribution is more concentrated around a certain point.

The less structure a random variable has, the higher is its unpredictability and therefore the higher its entropy. For random variables with equal variance, the variable with a gaussian distribution will naturally have the highest entropy, since it is the distribution with the least rate of predictability when a random draw is considered. This quality makes it a good candidate for a measure of nongaussianity in the context of independent component analysis.

In the case of FASTICA, the concept of negentropy is used, which can be defined as the difference in the level of entropy between a random variable y and a normally distributed random variable of equal variance y_{Gauss}. Negentropy is mathematically defined as:

$$J(y) = H(y_{Gauss}) - H(y) \tag{2.6}$$

An objective function can easily be constructed using this criterion based on which gradient decent algorithms can be built in order to estimate the components efficiently. For a more detailed description, see Hyvärinen, Karhunen, and Oja (2001) who provides ample detail and examples.

A related technique is based on the minimisation of Mutual Information and is a natural measure of dependence between random variables and takes into account the entire dependency structure.

$$I(y_1, y_2, \dots, y_n) = \sum_{i=1}^{n} H(y_i) - H(y) \tag{2.7}$$

It can be shown that the estimation of ICA by minimising mutual information is equivalent to the maximisation of the sum of nongaussianities of the estimates of the independent components. However, the estimation procedure applied for both this technique and FASTICA discussed above exceed the scope of this review and can be consulted in detail in Hyvärinen, Karhunen and Oja (2001) or one of the other references on the subject.

Further independent component estimation techniques based on information theory have been introduced in the past. Cardoso (1998) introduced JADE, and Learned-Miller and Fisher (2003) introduced RADICAL, both are based on the Kullback-Leibler divergence to quantify the divergence between the joint distribution and the product of independent marginal distributions.

The estimation method, which does retain our attention, is based on the maximisation of the likelihood function. It leads to an estimation technique referred to as Infomax and introduced by Bell and Sejnowski (1995) and Nadal and Praga (1996). To guarantee maximal independence of the components, the estimation must be evaluated under the most general assumptions possible. Let $f(R_i)$ be the probability density function of the returns vector of the considered assets. The density can be formulated as follows:

$$f(R_i) = |\det \mathrm{B}| f(S) = |\det \mathrm{B}| \prod_{j=1}^{n} f_j(S_j) \tag{2.8}$$

where $B = A^{-1}$, with $B = (b_1, \ldots, b_n)^T$, and $S_j = b_j R$. Given a sample of Z observations $R_i(1), R_i(2), \ldots, R_i(Z)$, of the vector R_i, $f_j(S_j)$ represent the densities of the independent components. Depending upon the context, constructing the likelihood function can be complicated; however, in practice it is often assumed that the Z observations of the return vector R_i are independent of each other as serial correlation can generally be considered zero. The Z observations of the returns can therefore be considered independent observations of the random variables $R_1, \ldots, R_n$. The likelihood function is then determined as a product of the densities evaluated at the Z points:

$$L(B) = \prod_{z=1}^{Z} \prod_{j=1}^{n} f_j(b_j^T R_i(z)) \, [|\det \mathrm{B}|] \tag{2.9}$$

Generally, it is often more practical to rewrite the relation as a log-likelihood:

$$log\, L(B) = \sum_{z=1}^{Z} \sum_{j=1}^{n} log\, f_j(b_j^T R(z)) + Z\, log|\det \mathrm{B}| \tag{2.10}$$

At this point the densities should be selected parametrically, since non-parametric estimation would involve the estimation of vast numbers of parameters, which is not generally feasible. Hyvärinen, Karhunen and Oja (2001) provide a criterion, which permits the selection of simple density families without affecting the local consistency of the Maximum Likelihood estimator when the densities respect the following criterion.

$$E[S_j g_j(S_j) - g_j{}'(S_j)] > 0 \tag{2.11}$$

Any density $g_j(S_j)$ respecting this criterion is in principle a suitable choice. (Hyvärinen, Karhunen, & Oja, 2001) propose two families, a super- and a subgaussian family of densities. Both are relatively simple densities, as small errors in the choice of density will not impact on the local consistency of the estimator. It is in fact sufficient that the densities lie in the same half of the probability space as the true density of the independent component would. The problem therefore becomes a binary one in the selection of the correct density.

Hyvärinen, Karhunen and Oja (2001) propose the two following density functions, the first one in the supergaussian case (indicated with a + sign), the second in the subgaussian case (indicated with a – sign):

$$\log f_i^{+}(S_i) = \beta_1 - 2\log\cosh(S_i) \tag{2.12}$$

$$\log f_i^{-}(S_i) = \beta_2 - \left[\frac{S_i^2}{2} - \log\cosh(S_i)\right] \tag{2.13}$$

The main motivation behind choosing these functions is that the supergaussian function is close to the Laplacian function, while the subgaussian function is in fact a flattened gaussian distribution, centred and normalised, which is the common assumption in ICA. But these functions are, not unique and can be replaced with other functions that respect the local consistency criterion defined in (2.11).

The estimation method described above has led to a very popular algorithm for the estimation of independent components, known as Infomax. Later adaptation and extension of the method improved it further. It can also be verified that likelihood maximisation is related to the JADE and FASTICA methods, even though this goes beyond the scope of this overview.

Several other methods have been developed to estimate independent components. The unifying feature is the estimation of the independent components through a specific objective function. What sets them apart is the speed and accuracy of convergence to the real independent components. For an excellent review we suggest consulting Hyvärinen (2013), which gives a comprehensive overview of recent progress in the field.

2.2.3 Properties

Several properties of the independent components should be highlighted at this stage. Three properties in particular are of importance. The independent components are estimated under the most general assumptions possible; this implies that the estimation can be carried out "blind", and implies that both the mixing matrix and the independent components are estimated from the sole observation of the signal, or asset prices.

Two properties of Independent Component Analysis, which largely influence the interpretation, and utilisation of the results are noteworthy at this point. The first property, as pointed out by Tong, Liu, Soon and Huang (1991) is that the exact order of the independent components cannot be determined. Formally, it can be shown that a permutation matrix P exists, such that $BA = PD$, with D as a diagonal scaling matrix. This implies that $R = AP^{-1}PS$. The elements of PS are the original independent variables S_i, of which the order has been changed. The result of this relation is that the order of the independent components cannot be determined.

The second property is that the variance of the independent components also cannot be determined. As the mixing process is completely unknown, A and S are both estimated through Independent Component Analysis. Multiplying S_i with a scalar and dividing the according rows of A by the same scalar would have no influence. Hyvärinen, Karhunen and Oja (2001) therefore states that the magnitudes of the independent components may as well be fixed beforehand, as they are random variables, which leads to fixing the variance of the independent components equal to one $E[S_i^2] = 1$.

The consequences of the two properties mentioned above are twofold. On the one hand ICA cannot be used for dimensionality reduction, as the order and therefore the importance of the components cannot be uniquely defined. Secondly, each of the independent components cannot be given a specific interpretation apart from being the fundamental drivers or building blocks of the data. The complete statistical independence between the components compensates for these shortcomings. The decomposition is therefore a cross-sectional one, where the independent components can be seen as the marginal distributions of the higher order dependencies existing between the random variables. Several applications in finance, such as Back and Weigend (1997) and the present one have shown this to be the case.

2.2.4 Further extensions

Independent Component Analysis as presented above is only the base case (also referred to as noiseless ICA) and several extensions have been developed. Two main extensions are noisy ICA and nonlinear ICA. In addition to these two extensions, ICA has also been adapted to suit the specific needs of some applications, such as spatiotemporal ICA and topographic ICA. However, the first two, given possible future work involving these techniques, are our present topics of analysis.

Let us first consider noisy ICA. The formulation of ICA chosen above and in the rest of this work is the noiseless version which implies that the decomposition of the variables R_i into their independent components is exact. The choice is based on the fact that good results were obtained before using the noiseless formulation and because ICA will be used as a change of variables rather than as a filtering technique.

Noisy ICA assumes that decomposition is not exact and part of the information stored in the random variables R_i is in fact noise and not essential to the identification of the true sources. It can be expressed as below:

$$R_i = \sum_{j=1}^{n} a_{ij} S_j + \varepsilon \text{, where } a_{ij} \text{ are real coefficients, } i, j = 1, \dots, n \qquad (2.14)$$

The noise component ε_j is considered to be independent of the independent components and Gaussian. The identification of the mixing matrix happens using the same assumption as in the base case ICA, implying full statistical independence and nongaussianity as a measure to extract the components. However, contrary to the noiseless formulation, the invertability is no longer possible. As a consequence, several denoising methods are considered in the estimation.

The second extension, which could be of future importance to our present work, is nonlinear ICA. It is in fact a generalisation of linear ICA, the method described above. Formally speaking, a function f can be defined from $\mathbb{R}^n \to \mathbb{R}^n$ such that:

$$R_i = f(AS) \tag{2.15}$$

Estimation techniques to retrieve the independent components have been proposed in various forms, most notably by Burel (1992), Deco and Bauer (1995), Deco and Obradovic, (1996), Lee, Bell, and Lambert (1997), Pajunen and Karhunen (1997) and Hyvärinen and Pajunen (1999).

It can be shown that a unique solution can be found except for some degenerate cases. Generally speaking, the nonlinear decomposition allows for deeper understanding of the structure of the data where linear mixtures are inadequate.

2.3 Conclusion

The primary focus of this chapter has been to set the stage for the principal derivations, which are the main focus of the next three chapters. A key component of these derivations is a recent mathematical technique known as Independent Component Analysis (ICA), a technique which performs Blind Source Separation of a set of mixed random variables.

Independent Component Analysis has had a relatively successful history since its introduction in 1987, 1991 and 1994. It has found applications in medical imaging, telecommunication, statistical signal decomposition and ultimately also in finance. In each of the cases the technique was shown to provide useful insights into the evolution of the analysed data.

The key feature of ICA is that it decomposes the data blindly, thereby implying that the estimation is performed under the most general conditions possible. Consequently, maximal independence can be guaranteed. Using that feature and the linearity of the decomposition will be of high importance in the derivations that follow in chapters 4, 5 and 6.

3

Portfolio Choice with Independent Components: The Case of the CARA Investor

3.1 Introduction

The previous chapters have shown how financial economic theory dictates that a portfolio can only be optimal when it maximises the investor's expected utility. This formulation is elegant in that it permits the expression of the investor's preferences with regard to both risk and expected return. However, the mathematical formulation of the optimization problem makes solving it analytically a challenge unless several restrictive hypotheses are introduced. In the context of single-period investing Markowitz (1952) suggests approximating the utility curve, ignoring in this way the higher moments of the asset returns, which have proven to be of high importance, see Mandelbrot (1963), Fama (1963), Blattberg and Gonedes (1974), Kon (1984) Loretan and Phillips (1944), Longin (1996). Several important advances have subsequently been made to include the probabilistic and stochastic characteristics of the asset returns in portfolio choice discussed in previous chapters of this thesis. But none of these advances has so far led to a full analytical solution for the single-period portfolio optimisation problem.

The present chapter is dedicated to the first step in the derivation of a fully analytical solution for a utility maximising expected utility portfolio, an alternative method we present in this work. The derivation concerns the particular case of a single-period CARA investor for which a full-scale analytical solution can be found to the maximisation of the expected utility. This solution has many attractive features starting with the fact that it does not require additional assumptions or approximations, see Levy (1969), Samuelson (1969), Samuelson (1969), Jean (1971), Levy and Sarnat (1984), Jondeau and Rockinger (2006), Harvey, Liechty, Liechty and Müller (2010). As a consequence, it includes all higher moments of the asset return distributions as well as all properties of the utility curve. Central to the approach is the decomposition of the asset space using Independent Component Analysis, or ICA. The linear decomposition of the asset space allows for the full factorization of the optimisation problem. The multidimensional optimisation of the expected utility is therefore simplified to a one-dimensional problem.

In summary, our proposed method therefore has three important advantages. It includes all higher moments of the asset distribution; it avoids introducing either approximations of the utility or the distributions of the assets; and finally it works directly from the asset returns, the optimisation steers clear of parameter uncertainty related to the estimation of the input parameters Klein and Bawa, (1976), Brown and Weinstein (1978), Jobson and Korkie (1981), Black and Litterman (1990), Jorion (1985).

3.2 Portfolio Selection

In order to provide a clear derivation of the model we start from first principles. Let us assume that an investor holds an initial wealth of W_0, arbitrarily fixed to 1 at the beginning of the period. The investor seeks to invest in a portfolio defined by a set of preferences represented by the maximisation of the investor's utility.

The end-of-period wealth is denoted, W, from interval $I \subset \mathbb{R}$ to $\mathbb{R}$ and a von Neumann-Morgenstern utility function $U(.)$ is defined over W. There are n risky assets with return vector $R = (R_1, \dots, R_n)^T$ and joint cumulative distribution $F(R_1, \dots, R_n)$. The investor aims to maximize the expected utility of the end-of-period wealth $U(W)$, which is given by

$W = 1 + r_p$, with $r_p = \alpha^T R$, where the vector $\alpha = (\alpha_1, \dots, \alpha_n)^T$ represents the fractions of wealth invested in each of the various risky assets. For the individual equilibrium we assume that the investor does not have access to a riskless asset. The portfolio weights need to satisfy two conditions: their sum must be equal to one ($\sum_{i=1}^{n} \alpha_i = 1$), and they must be non-negative, thereby prohibiting the possibility of short selling.

Formally, the portfolio allocation is obtained by solving the following standard maximisation problem:

$$E[U(W)] = \int U(W)\, f(W) dW \tag{3.1}$$

where $f(W)$ is the probability density function of the end-of-period wealth of the portfolio. Using the transformation invariance property of the expectation, the investor maximises:

$$\begin{cases} E[U(W)] = E[U(1 + \alpha^T R)] = \int \dots \int U(1 + \sum_{i=1}^{n} \alpha_i R_i) dF(R_1, \dots, R_n) \\ \qquad s.t. \sum_{i=1}^{n} \alpha_i = 1 \;\; and\; \alpha_i \geq 0, for\; i = 1, \dots, n \end{cases} \tag{3.2}$$

In order to solve the optimization problem analytically, we follow three steps, which will be described in detail in sections 3.2.1, 3.2.2 and 3.2.3; our objective here is to transform the complex integration into a simplified form. We first substitute the dependent variables $R_1, \dots, R_n$ with an equal number of independent variables $S_1, \dots, S_n$, so that the joint asset returns distribution is reduced to a product of n one-dimensional distributions (section 3.2.1). We next select a utility function in order to define a complete factorisation of the optimization problem and in so doing simplify the complex multiple integral to a product of n one-dimensional integrals (section 3.2.2). We then estimate the densities for the independent variables $S_1, \dots, S_n$ to obtain the final analytical solution of the problem (section 3.2.3).

3.2.1 Change of variables: Independent Component Analysis (ICA)

To perform the change of variables, we implement an Independent Component Analysis (ICA). Independent Component Analysis, as explained in Hyvärinen, Karhunen and Oja (2001), defines a generative model which describes how a set of n observed random variables, in our case the returns of n assets $R_1, \dots, R_n$, are generated by an equal number of independent components $S_1, \dots, S_n$ in the following way:

$$R_i = \sum_{j=1}^{n} a_{ij} S_j \text{, where } a_{ij} \text{ are real coefficients, } i,j = 1, \dots, n \tag{3.3}$$

By considering the decomposition of the stock returns, the optimisation problem defined in equation (4.3) can now be simplified and rewritten as follows:

$$\begin{cases} E[U(W)] = \\ \int \dots \int U\left[1 + \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{n} a_{ij} S_j \right)\right] f_1(S_1) f_2(S_2) \dots f_n(S_n) |J_B| dS_1 dS_2 \dots dS_n \\ s.t. \sum_{i=1}^{n} \alpha_i = 1 \;\; and \; \alpha_i \geq 0, for \; i = 1, \dots, n \end{cases} \tag{3.4}$$

where $|J_B|$ is the Jacobian of the matrix B, which is the inverse of matrix A. The distribution functions of the independent components are given by $f_1(S_1) f_2(S_2) \dots f_n(S_n)$.

3.2.2 The utility function

We select the Constant Absolute Risk Aversion (CARA) utility function in this case to represent investor preferences. The CARA utility function is part of a group of commonly used utility functions that allow for the generalisation of the optimisation problem to reach market equilibrium. It is defined as follows:

$$U(W) = -exp\,(-\vartheta W) \tag{3.5}$$

where ϑ measures the investor's risk aversion coefficient.

Given the exponential nature of the CARA utility function, equation (4.4) can now be formulated:

$$\begin{cases} E[U(W)] = \\ |J_B| \int U\left(1 + \sum_{i=1}^{n} \alpha_i a_{i1} S_1\right) f_1(S_1) dS_1 \dots \int U\left(1 + \sum_{i=1}^{n} \alpha_i a_{in} S_n\right) f_n(S_n) dS_n \\ s.t. \sum_{i=1}^{n} \alpha_i = 1 \;\; and \;\; \alpha_i \geq 0, for\; i = 1, \dots, n \end{cases} \tag{3.6}$$

The problem, which was initially identified by a complex multiple integral, has been simplified and now takes the form of a product of n one-dimensional integrals. The problem thus simplified allows us to define the densities for the independent components, evaluate each of the integrals, and then solve the optimisation problem.

3.2.3 The densities of the independent components and solution of the n one-dimensional integrals

The definition of the densities for the independent components is intrinsically linked to the estimation of the components. The estimation must be implemented under the most general assumptions in order to guarantee maximum independence of the components. Following Hyvärinen, Karhunen and Oja (2001), we apply maximum likelihood estimation to find the independent components. In chapter 2 we showed that this leads to the selection of two simple families of distributions, a supergaussian and a subgaussian density to cover the entire probability space.

Several existing families of nongaussian densities do not affect the local consistency of the Maximum Likelihood estimator as was shown in chapter 2. In our case we select super- and subgaussian densities so that the entire probability space may be covered:

$$g_j(S_j)_{sub} = exp\ (v_j)\left(\frac{1}{2}exp\left(-\frac{S_j^2}{2}\right)\left(exp(S_j) + exp(-S_j)\right)\right) \tag{3.7}$$

$$g_j(S_j)_{super} = \frac{1}{2}sech\left(\frac{\pi}{2}S_j\right) \tag{3.8}$$

The supergaussian density is in fact the hyperbolic secant distribution; by being super- and subgaussian, they provide the necessary nongaussianity condition to estimate the independent components. The selected densities are symmetrical (a standard hypothesis in ICA estimation); moreover, by whitening the asset returns data they become centred and have unit variance.

Let S_k and S_l be two independent components obtained from the original data set of asset returns through ICA, with $1 \leq k, l \leq n, k \neq l,$ and S_k has a subgaussian distribution, whereas S_l has a supergaussian distribution. Equation (3.6) can now be rewritten as follows:

$$
\begin{gathered}
E[U(W)] = \\
|J_B| \dots \int_{-\infty}^{+\infty} -exp(-\vartheta) \frac{exp(v_k)}{2} \left[exp\left(\frac{-{S_k}^2}{2} + S_k - \vartheta \left(\sum_{i=1}^{n} \alpha_i a_{ik} S_k \right) \right) + \right. \\
\left. exp\left(\frac{-{S_k}^2}{2} + S_k - \vartheta \left(\sum_{i=1}^{n} \alpha_i a_{ik} S_k \right) \right) \right] dS_k \dots \\
\dots \int_{-\infty}^{+\infty} -exp(-\vartheta) \frac{1}{2} \left[exp\left(-\vartheta \left(\sum_{i=1}^{n} \alpha_i a_{il} S_l \right) \right) sech\left(\frac{\pi}{2} S_l \right) \right] dS_l \dots \\
s.t. \sum_{i=1}^{n} \alpha_i = 1 \;\; and \; \alpha_i \geq 0, for \; i = 1, \dots, n
\end{gathered}
\tag{3.9}
$$

which can be represented in a simplified form by:

$$
E[U(W)] = \prod_k Sub_k \cdot \prod_l Super_l \tag{3.10}
$$

The subgaussian side has a closed form and a converging solution known as the gaussian integral[3] and in the context of the optimization problem, the solution can be written in the following way:

[3] The standard solution to the Gaussian integral can be written so that the parameters ψ, φ and η are used to present a standard case of the integral. $\int_{-\infty}^{+\infty} \exp(-\psi x^2 + \varphi x + \eta)\, dx = \sqrt{\frac{\pi}{\psi}} \exp\left(\frac{\phi^2}{4\phi} + \eta\right)$

$$\int_{-\infty}^{+\infty} -exp(-\vartheta)\frac{exp(v_k)}{2}\left[exp\left(\frac{-{S_k}^2}{2}+S_k-\vartheta\left(\sum_{i=1}^{n}\alpha_i a_{ik}S_k\right)\right)\right.$$
$$\left.+\,exp\left(\frac{-{S_k}^2}{2}+S_k-\vartheta\left(\sum_{i=1}^{n}\alpha_i a_{ik}S_k\right)\right)\right]dS_k$$
$$=-exp(-\vartheta)\frac{exp(v_k)}{2}\left[\sqrt{2\pi}\,exp\left(\frac{\left(1-\vartheta\sum_{i=1}^{n}\alpha_i a_{ik}\right)^2}{2}\right)\right.$$
$$\left.+\sqrt{2\pi}\,exp\left(\frac{-\left(1+\vartheta\sum_{i=1}^{n}\alpha_i a_{ik}\right)^2}{2}\right)\right] \tag{3.11}$$

The supergaussian side is more complicated but when we use simplified notation and a change of variables, the integral can be rewritten as:

$$-exp(-\vartheta)\int_{-\infty}^{+\infty}\frac{1}{2}\left[exp\left(-\vartheta\left(\sum_{i=1}^{n}\alpha_i a_{il}S_l\right)\right)sech\left(\frac{\pi}{2}(S_l)\right)\right]dS_l$$
$$=-exp(-\vartheta)\frac{\pi}{2}\int_{-\infty}^{+\infty}[exp(-b_l^*S_l^*)\,sech(S_l^*)]dS_l \tag{3.12}$$

$$b_l^*=\frac{2}{\pi}b_l,\qquad b_l=\vartheta\left(\sum_{i=1}^{n}\alpha_i a_{il}\right),\qquad S_l^*=\frac{\pi}{2}S_l,\qquad \frac{dS_l}{dS_l^*}=\frac{\pi}{2}$$

The right-hand side of equation (3.12) takes the shape of a two-sided Laplace transform as defined by Eurle (1785).[4] The solution for the integral in equation (3.12) can now be found:

4 The two-sided Laplace transform for the hyperbolic secant as defined by (Eurler, 1785) and a recent translation of the paper by (Gélinas, 2012) takes the following shape: $\int_{-\infty}^{+\infty}[exp(-wt)\,sech(t)]dt=\frac{2\pi\,sin\left(\frac{\pi}{2}w\right)}{sin(\pi w)}$, given $|w|<1$.

$$-exp(-\vartheta)\int_{-\infty}^{+\infty}\frac{1}{2}\left[exp\left(-\vartheta\left(\sum_{i=1}^{n}\alpha_i a_{il}S_l\right)\right)sech\left(\frac{\pi}{2}S_l\right)\right]dS_l$$
$$= -\frac{\pi^2\,exp(-\vartheta)}{4}\left[sec\left(\vartheta\left(\sum_{i=1}^{n}\alpha_i a_{il}\right)\right)\right] \tag{3.13}$$

with $|\vartheta(\sum_{i=1}^{n}\alpha_i a_{il})| < 1$. This latter condition is a realistic one. In fact, since the S_l are distributed with variance equal to 1, and the variance of the asset returns is well below 1, the scaling is carried out through the values of the a_{il}. Therefore, over the short to medium term investment periods, this condition is fulfilled, even for large risk aversion coefficients.

3.2.4 The optimal portfolio for individual and market equilibrium

At this stage we can draw together the various elements and return to the initial optimisation problem defined as:

$$\begin{cases} E[U(W)] = \\ |J_B|\left[\dots\left[-\frac{exp(-\vartheta)exp(v_k)}{2}\begin{pmatrix}\sqrt{2\pi}\,exp\left(\frac{\left(1-\vartheta\sum_{i=1}^{n}\alpha_i a_{ik}\right)^2}{2}\right)+ \\ \sqrt{2\pi}\,exp\left(\frac{-\left(1+\vartheta\sum_{i=1}^{n}\alpha_i a_{ik}\right)^2}{2}\right)\end{pmatrix}\right]\dots\right. \\ \left.\dots\left[-\frac{\pi^2\,exp(-\vartheta)}{4}\left(sec\left(\vartheta\left(\sum_{i=1}^{n}\alpha_i a_{il}\right)\right)\right)\right]\dots\right] \\ s.t.\sum_{i=1}^{n}\alpha_i = 1 \;\; and\; \alpha_i \geq 0, for\; i = 1,\dots,n \end{cases} \tag{3.14}$$

A problem that began as a complex multiple integral has now been transformed into a tractable product of closed form solutions of single integrals.

The optimisation shown above leads to individual equilibrium for the investor. However, in order to examine different market conditions and evaluate different asset classes we need to shift from the individual equilibrium to a market equilibrium model. By so doing we separate the portfolio following Cass and Stiglitz (1970), where maximisation of the investor's utility is achieved by investing in two funds: a riskless asset and a market portfolio. The condition, which ensures portfolio separation, is that all investors have either hyperbolic or exponential utility. Our model has been derived for a CARA utility, so this condition is satisfied.

The investor's end-of-period wealth is given by $W = 1 + \alpha_0 R_f + \alpha_1 R_m$, where R_f is the return on a risk-free asset and R_m is the return on the market portfolio. Assuming the independent components: S_0 to be subgaussian and S_1 to be supergaussian, we obtain the following relation for market equilibrium:

$$\left\{\begin{array}{l} E[U(W)] = E\left[U\left(1+\alpha_0 R_f + \alpha_1 R_m\right)\right] = \\ |J_B| \left[\left[-\frac{\exp(-\vartheta_m)exp(v)}{2}\left(\sqrt{2\pi}\,exp\left(\frac{\left(1-\vartheta_m \sum_{i=0}^{1}\alpha_i a_{i0}\right)^2}{2}\right)\right.\right.\right. \\ \left.\left. +\sqrt{2\pi}\,exp\left(\frac{-\left(1+\vartheta_m \sum_{i=0}^{1}\alpha_i a_{i0}\right)^2}{2}\right)\right)\right] \ldots \\ \left.\left[-\frac{\pi^2\,exp(-\vartheta_m)}{4}\left(sec\left(\vartheta_m\left(\sum_{i=0}^{1}\alpha_i a_{i1}\right)\right)\right)\right]\right] \\ s.t..\ \sum_{i=0}^{1}\alpha_i = 1\ \ and\ \alpha_0, \alpha_1 \geq 0 \end{array}\right. \tag{3.15}$$

where ϑ_m is the risk aversion.

Further investigation into the presented results reveals that equations (3.14) and (3.15) can now be estimated. The optimal portfolio is defined by using the first-order conditions (FOCs) of the optimisation problem:

$$\frac{\partial E[U(W)]}{\partial \alpha_i} = \lambda_i \tag{3.16}$$

where the left-hand side indicates the first derivative of U with respect to the i^{th} asset, and λ_i is the Lagrange multiplier associated with the optimisation problem. The solution of the optimal portfolio has now taken the shape of a function of the independent components and the mixing coefficients, which can be estimated by most optimisation software, in our case Matlab's optimisation functions.

Let us consider an illustrative example of three portfolios constructed from a two-asset space. We assume that asset 1 consists of 75% of independent component 1, and 25% of component 2, following the decomposition through ICA. Meanwhile, asset 2 consists of 25% of independent component 1, and 75% of component 2. Table 3-1 summarises the three portfolios through the use of three distinct risk-aversion coefficients.

Table 3-1: Case overview of ICA-based portfolios

The present table illustrates the case of a portfolio consisting of 2 assets decomposed into 2 independent components. It systematically proceeds through the cases and interpretations linked to various degrees of dependence between the two assets. Each of the cases is described in detail.

	ICA Weighting Matrix		**Portfolio Composition**
	Weight for Component 1	**Weights for Component 2**	**Interpretation**
Weights for Asset 1	0.75	0.25	After decomposing a two-asset based sample through ICA, it turns ou that asset 1 consists for 75% of component 1 and 25% of component 2. The reverse is true for asset 2
Weights for Asset 2	0.25	0.75	
Risk Aversion	**Portfolio Weight for the Assets**		
ϑ	**α1**	**α2**	**Interpretation**
High	1.00	0.00	Also further down: Given the weighting and the mixing matrix, the portfolio will depend on component 1 for 75% and for 25% on component 2. These values correspond to the weights of the inedependent components in asset 1
Middle	0.75	0.25	The weight of asset 1 is reduced and thereby the importance of component 1. Component 2 takes up the freed space. The portfolio depends on component 1 for 62.5% and for the remainder, on component 2
Low	0.5	0.5	50% of asset 1 is included in the portfolio which implies that 37.5% of component 1 and 18.75% of component 2. To this we add 12.5% of component 1 and 37.5% of component 2. The final result is a portfolio equally dependent on both components

The risky portfolio is based exclusively on one asset and is therefore exposed to all idiosyncratic risk of that particular asset and does not benefit from diversification. The risk-averse portfolio is equally weighted between the assets. When we examine the importance of the components in each portfolio, it is apparent that the most risky portfolio opts for maximum exposure to certain co-moments of the joint distribution of the asset space (in this case captured by the first independent component). Conversely, the risk-averse portfolio attempts to eliminate as much of the co-moment risk as possible by granting both components the same importance in the portfolio.

3.3 Concerns regarding the model

Even though the presented model looks mathematically promising, it is suffering from two potentially limiting weaknesses. These weaknesses are on the one hand its mathematical formulation, which includes the secant function, and on the other hand the limitations imposed by the CARA utility function.

The portfolio model presented in equation (3.14) is heavily dependent on the behaviour of the secant function. This function has an asymptotic behaviour following the inverse of the cosine function. The consequence is that many very distinct portfolio compositions will lead to values of the function that are very close to each other. Software packages that are not prepared for retrieving such minimal differences between possible local and global minima will not retrieve the right solution. Additionally, if the estimation of the independent components presents some small degree of variation, it could lead to sub-optimal portfolio compositions.

The second drawback relates to the CARA utility function itself. Even if this function has had noteworthy success and influence in the economic and financial world, it is not considered as the benchmark utility function following significant empirical work. A portfolio model, which, through its mathematical structure, does not allow for a change in the utility function, consequently will not be a sufficiently valuable addition to the state of the art.

Both criticisms are important and have therefore led to the decision to continue the research and derive a second model, which is presented in the next chapter.

3.4 Conclusion

The present chapter has derived a fully analytical solution to a full-scale optimisation for an expected utility maximising investor with CARA preferences. In doing so, we have shown how approximation commonly made in order to ensure solvability of the multiple integrals is not required for CARA investors. This solution is in contrast to recent models such as Jondeau and Rockinger (2006), which limit the number of moments to the first four and discard the error term when using a Taylor series approximation.

The solution to the portfolio choice problem is of course only relevant for the special case of a CARA investor. In subsequent chapters applications of the methodology will also prove its validity in terms of out-of-sample out performance for that specific case. These applications and tests will show how Independent Component Analysis is fully able to capture higher order dependencies that are not captured by EV.

In order for this approach to be valid for portfolio choice in general and more specifically also to asset classes which have particular features, such as infrastructure, it is necessary for the portfolio choice strategy to be generalised. This task will be the objective of the next chapter.

4
Portfolio Choice with Independent Components: A Generalized Case

4.1 Introduction

In chapter 3 we derived a first full scale analytical solution for an expected utility maximising CARA investor by using a change of variables through Independent Component Analysis to find closed form solutions to the multiple integrals. ICA in this case has allowed for the full factorisation of the problem by providing a cross-sectional decomposition of the asset space. In this respect, ICA has effectively separated the variables' dependency structure, which would otherwise remain inside the optimisation in the mixing matrix.

The step described above is fundamental, as it allows for a single-period framework to be well adapted for longer-term investments, where rebalancing the portfolio is not possible or desirable, as in the example of infrastructure investment and infrastructure operators. Rather than focussing on the stochastic nature of the asset return data, and thereby redefining the probability density function on a constant basis, emphasis is placed on finding an accurate description of the interdependence structure of the asset returns. That structure, even if the decomposition is stationary, allows for a thorough understanding of the actual drivers of the returns, thereby providing the necessary longer-term optimisation tools.

The objective of the present chapter is to go beyond the model presented in chapter 3 and truly generalise the idea of a higher moment single period expected utility maximising model for all commonly used von Neumann-Morgenstern utility functions. The model is therefore in some sense a generalisation of chapter 3 in that it keeps the advantages of using ICA as an instrument to simplify the mathematical formulation without imposing restrictive hypotheses to achieve solvability. The result is therefore a single-period model, which includes all higher moments and avoids other shortcomings of traditional EV, such as parameter uncertainty.

4.2 Generalized portfolio choice with ICA

The starting point for our derivation is the classical financial economic assumption of investor expected utility maximisation for single-period investments. We assume that an investor holds an initial wealth of W_0, arbitrarily fixed to 1 at the beginning of the considered period. The end-of-period wealth is denoted, W, from interval $I \subset \mathbb{R}\ to\ \mathbb{R}$ and a von Neumann-Morgenstern utility function $U(.)$ is defined over W, which defines the investors preferences. The asset space from which the investor will choose its optimal portfolio, consists of n risky infrastructure assets with return vector $R = (R_1, \dots, R_n)^T$ and joint cumulative distribution $F(R_1, \dots, R_n)$. End-of-period wealth can be represented by $W = 1 + r_p$, with $r_p = \alpha^T R$, where the vector $\alpha = (\alpha_1, \dots, \alpha_n)^T$ represents the fractions of wealth invested in each of the various risky assets. For the individual equilibrium we assume that the investor does not have access to a riskless asset. The portfolio weights need to satisfy two conditions: their sum must be equal to one ($\sum_{i=1}^{n} \alpha_i = 1$) and they must be non-negative, thereby prohibiting the possibility of short selling.

Formally, the portfolio allocation is obtained by solving the following standard maximisation problem:

$$\begin{cases} \max_{\alpha_i} E[U(W)] = \int U(W)f(W)dW = \int \dots \int U(1 + \sum_{i=1}^{n} \alpha_i R_i)dF(R_1, \dots, R_n) \\ s.t. \sum_{i=1}^{n} \alpha_i = 1 \ and \ \alpha_i \geq 0, for \ i = 1, \dots, n \end{cases} \tag{4.1}$$

where $f(W)$ is the probability density function of the end-of-period wealth on the portfolio, which depends on the vector of weights α.

As presented by Jurczenko and Maillet (2006), which is the derivation we are following, the utility $U(W)$ of the investor can be expressed as a Taylor expansion if the utility function is arbitrarily and continuously differentiable in I. The function will be evaluated at $\bar{W} = E[W] = 1 + \alpha^T \mu$ and $\mu = E[R]$, for all $W \in I$:

$$U(W) = \sum_{k=0}^{N} \frac{U^{(k)}(\bar{W})(W - \bar{W})^k}{k!} + \mathcal{E}_{N+1}(W) \tag{4.2}$$

$\mathcal{E}_{N+1}(W)$ is the Lagrange remainder defined as:

$$\mathcal{E}_{N+1}(W) = \frac{U^{(N+1)}(\xi)}{(N+1)!} [W - \bar{W}]^{(N+1)} \tag{4.3}$$

where $\xi \in]W, E(W)[\ if \ W < E(W)$, or $\xi \in]E(W), W[\ if \ W > E(W)$, and $N \in \mathbb{N}^*$

If we now make the assumptions that the Nth order Taylor approximation of $U(W)$ around $E[W]$ converges towards $U(W)$ for $N \to \infty$; that the integral and summand operators are interchangeable, and that a distribution of the returns is uniquely determined by the moments which exist for all orders, we can rewrite equation (4.1) using the limit of N going towards infinity and taking the expected value at both sides:

$$E[U(W)] = \int_{-\infty}^{+\infty} \left\{ \lim_{N \to \infty} \left[\sum_{k=0}^{N} \frac{U^{(k)}(\bar{W})(W - \bar{W})^k}{k!} + \mathcal{E}_{N+1}(W) \right] \right\} dF(W) \tag{4.4}$$

$$= U[E(W)] + \frac{1}{2!}U''[E(W)]E[[W - E(W)]^2] + \frac{1}{3!}U'''[E(W)]E[[W - E(W)]^3]$$

$$+ \frac{1}{4!}U''''[E(W)]E\left[\left[r_p - E(W)\right]^4\right] + \cdots$$

$$+ \frac{1}{N!}U^{(N)}[E(W)]E[[W - E(W)]^N]$$

$$as \lim_{N \to \infty} \mathcal{E}_{N+1}(W) = 0$$

the infinite order Taylor approximation used is the following:

$$E[U(W)] = E\left[\sum_{k=0}^{\infty} \frac{U^{(k)}(\bar{W})(W - W)^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{U^{(k)}(\bar{W})}{k!} E[(W - \bar{W})^k] \tag{4.5}$$

This approximation only holds if three conditions are satisfied as stated by Jurczenko and Maillet (2006) for the Taylor series approximation of the expected utility to converge to its actual value if the number of terms in the approximation is infinite. 1) The utility function should be an analytic function at E[W]. Its realised returns leading to the end-of-period wealth should remain inside the absolute convergence interval of the Taylor series expansion of the considered utility function, (see Tsiang (1972) Loistl (1976) Lhabitant (1997)). 2) Uniform convergence towards U(W) should be attained whereby the summand operators are interchangeable with the integral in the expected utility function (see Loistl (1976), Lhabitant (1997) and Christensen and Christensen (2004)). 3) The Hamburger (1920) moment problem should be dealt with, implying the existence and uniqueness of a continuous positive distribution function of the returns and therefore the end-of-period wealth given a set of non-centred moments.

All three conditions are examined in detail by Jurczenko and Maillet (2006). They show in the case of the first condition that absolute convergence can be guaranteed for the

Taylor expansion of the utility function $U(W)$ around $E[W]$ if the realisations of the random variable W belong to an open interval J defined as:

$$|W - E(W)| < \zeta \text{ with}$$
$$\zeta = \lim_{N\to\infty} \left| \frac{(N+1)!\, U^{(N)}[E(W)]}{N!\, U^{(N+1)}[E(W)]} \right| \tag{4.6}$$

ζ is a positive constant which corresponds to the radius of convergence of the Taylor series expansion of the utility function around $E[W]$ with $N \in \mathbb{N}$. This convergence condition implies that for power and logarithmic utility functions a restriction on the wealth range is necessary of $0 < W < 2E(W)$.

Uniform convergence, which is the second condition set out by Jurczenko and Maillet (2006), requires shrinkage of the interval of absolute convergence:

$$|W - E(W)| \leq \zeta^*$$
$$\zeta^* \in]0, \zeta[\tag{4.7}$$

The third condition, the Hamburger moment problem, is of lesser concern here as we are only considering centred moments in our case. The generalisation to non-centred moments would require satisfying this condition too.

At this point the optimisation problem in equation (4.1) has been simplified to the infinite Taylor approximation in equation (4.5). Using Independent Component Analysis a cross-sectional decomposition of the returns of n assets $R_1, \dots, R_n$, into an equal number of independent components $S_1, \dots, S_n$ is performed. Taking the components into account:

$$1 + r_p = 1 + \sum_{i=1}^{n} \alpha_i R_i = 1 + \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{n} a_{ij} S_j \right) \tag{4.8}$$

and substituting definition (4.8) into equation (4.5) leads to the following simplification of the expected utility:

$$E[U(W)] = E\left[\sum_{k=0}^{\infty} \frac{U^{(k)}(\bar{W})(W-\bar{W})^k}{k!}\right]$$

$$= \sum_{k=0}^{\infty} \frac{U^{(k)}\left(E\left[\sum_{i=1}^{n} \alpha_i(\sum_{j=1}^{n} a_{ij}S_j)\right]\right)}{k!} E\left[\left(\sum_{i=1}^{n} \alpha_i(\sum_{j=1}^{n} a_{ij}S_j) - E\left[\sum_{i=1}^{n} \alpha_i(\sum_{j=1}^{n} a_{ij}S_j)\right]\right)^k\right] \tag{4.9}$$

The problem has now been rewritten in such a fashion that any higher order dependency between asset returns are captured by the elements of the mixing matrix, while the remaining random variables are independent.

At this juncture it is worthwhile to consider the estimation of the independent components. Several methods are available, however, we have selected the estimation through the maximization of the likelihood function, described in chapter 2. The method has the advantage of being highly intuitive. It provides the necessary nongaussianity criterion by qualifying the independent components as either super- or subgaussian. The use of this method leads to a significant simplification in the estimation without loss of consistency of the likelihood function.

A particularly interesting feature in our case is the nature of the selected distribution for the independent components. They belong to two families of simple nongaussian distributions, with unit variance, symmetry around the mean, and no independently defined higher moments. As a consequence only the following moments retain their importance in the definition of the expected utility:

$$\begin{cases} \mu_p = E[r_p] = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j] \\ \sigma_p^2 = E\left[\left(r_p-\mu_p\right)^2\right] = var(r_p) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2 var(S_j) \\ s_p^3 = E\left[\left(r_p-\boldsymbol{\mu_p}\right)^3\right] = skew(r_p) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^3 skew[(S_j)] \\ k_p^4 = E\left[\left(r_p-\mu_p\right)^4\right] = kurt(r_p) = \\ \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^4 kurt[(S_j)] + 6\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2 var(S_j)\right] \end{cases} \tag{4.10}$$

Of these moments, the variance of S_j equals 1, the skewness equals zero, and the kurtosis is a fixed value depending on the distribution of the S_j, as given in equation (4.11):

$$\begin{cases} E[S_j] = Constant \\ var(S_j) = 1 \\ skew[(S_j)] = 0 \\ kurt[(S_j)] = Constant \end{cases} \tag{4.11}$$

The expression for the expected utility can now be noticeably simplified:

$$\begin{aligned} E[U(W)] = U&\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right) \\ &+\frac{1}{2!}U^{(2)}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right)\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2\right) \\ &+\frac{1}{4!}U^{(4)}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right)\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^4 kurt[(S_j)]\right. \\ &\left.+6\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2\right]\right) \end{aligned} \tag{4.12}$$

Finding the optimal portfolio for an investor, given a certain utility function, can therefore be solved analytically without making any assumptions regarding the shape of the utility, the number of moments to be taken into account, or probability beliefs as stipulated by Markowitz (1952). The number of assumptions is limited to strictly a single one, that is the use of Independent Component Analysis as a useful decomposition of financial random variables.

4.3 Portfolio choice with ICA

Historically the utility functions of choice have been the von Neumann-Morgenstern functions, which are time separable with hyperbolic absolute risk aversion (HARA). In particular the logarithmic (decreasing absolute risk aversion - DARA), power (constant relative risk aversion - CRRA) and negative exponential (constant absolute risk aversion - CARA) utility functions. We will focus on these same functions in the context of the present chapter.

Using equation (4.12), we can now define the three final equations for the optimisation problem of expected utility using the three utility functions of choice:

$$\begin{cases} U_1(W) = -\ exp(-\theta W) \\ U_2(W) = Log(W) \\ U_3(W) = \dfrac{W^{(1-\eta)} - 1}{1-\eta} \end{cases} \tag{4.13}$$

The mathematical formulation of the optimisation is in all three cases very similar. Using the CARA utility we obtain equation (4.14) below:

$$E[U_1(W)]$$

$$= -exp\left(-\theta\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right)\right)$$

$$-\frac{\theta^2}{2}exp\left(-\theta\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right)\right)\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2\right)$$

$$-\frac{\theta^4}{24}exp\left(-\theta\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right)\right)\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^4 kurt[(S_j)]\right.$$

$$\left.+ 6\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2\right]\right) \tag{4.14}$$

The formulation of the optimisation using the log utility function is given in equation (4.15):

$$E[U_2(W)] = log\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right) + \frac{\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2\right)}{2\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right)}$$
$$-\frac{\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^4 kurt[(S_j)] + 6\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2\right]\right)}{24\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E[S_j]\right)} \tag{4.15}$$

Finally for the remaining power utility function the expected utility is expressed by equation (4.16):

$$E[U_3(W)] = \frac{\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E\left[S_j\right]\right)^{(1-\eta)} - 1}{1-\eta}$$
$$-\frac{\eta}{2}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E\left[S_j\right]\right)^{-\eta-1}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2\right)$$
$$+\frac{1}{24}(-\eta^3 + 3\eta^2 \tag{4.16}$$
$$+ 2\eta)\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)E\left[S_j\right]\right)^{-\eta-3}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^4 kurt\left[(S_j)\right]\right.$$
$$\left.+ 6\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\alpha_i a_{ij}\right)^2\right]\right)$$

To find the optimal asset allocation we maximise the lagrangian $\mathcal{L}$ in order to account for the constraints:

$$\mathcal{L} = E[U(W)] - \lambda\left(\sum_{i=1}^{n}\alpha_i - 1\right) \tag{4.17}$$

Taking the first order derivative of the Lagrangian with respect to the assets α_i, we find the n equations of the optimisation expressed as:

$$\frac{\partial\mathcal{L}}{\partial\alpha_i} = \frac{\partial E[U(W)]}{\partial\alpha_i} - \lambda = 0 \tag{4.18}$$

The maximisation in equation (4.18) can be performed analytically or numerically by most of the standard optimisation software packages. The results of these optimisations are discusses in the applications in the next chapters.

4.4 Conclusion

In the present chapter we have presented a generalisation to all commonly used utility functions of the use of ICA to obtain closed form solutions to the expected utility maximisation problem. The model has three very favourable properties, which make it an interesting addition to the financial literature. The first and foremost property is that the model is the first to present a full-scale optimisation of the expected utility maximising problem, for all utility functions, without limiting the number of moments taken into account. In later chapters where the model will be applied to empirical data, we will demonstrate how this aspect is of crucial importance.

The second main advantage is the use of ICA and its capacity to decompose the dependency structure of the asset space and make use of these dependencies in the allocation of assets to a portfolio. The first and second property highlighted in this conclusion has an intrinsic link in that the first one depends on the second one in the inclusion of all moments.

The third advantage or property, which should be highlighted, is that the portfolio is constructed directly from the input data that is the historical returns of the assets. Provided that sufficient returns are available, this allows the optimisation to work directly from the asset space and deduce any dependencies from this space without the prior interference of estimation techniques.

A last noteworthy element, is that the method does not introduce major innovation in terms of mathematical complexity. Apart from ICA, for which the literature on estimation (as well as computation of the components) is sufficiently large, the model is constructed using standard financial economic theories and methods. All the elements put together make the method an attractive addition to the literature.

The only wisdom we can hope to acquire
is the wisdom of humility: humility is endless.

Thomas Stearns Eliot

Part 3

Applications in Infrastructure Investment

5 Infrastructure Returns and Characteristics

5.1 Introduction

In part 2 of this work we derived a portfolio choice model based on Independent Component Analysis. Part three of the thesis is dedicated to the presentation of several testing results for the portfolio choice model discussed in chapter 4 using two infrastructure data sets. A very general sample of global infrastructure indexes will be considered, as these data form a good proxy for infrastructure returns and avoid sectoral or regional biases. Thereafter we consider a large sample of airport operators, as this infrastructure subsector is of particular interest for its particular return characteristics.

In order to increase clarity of the empirical tests conducted over the two next chapters, we will examine each of the data sets separately. A brief review of analyses of infrastructure returns shown in the literature is the topic of the next section.

5.2 Literature on infrastructure returns

Infrastructure has emerged relatively recently as a private asset, which is open to private investment. For this reason there are few in depth studies of the risk and return characteristics of infrastructure. One consequence of this lack of empirical work is that infrastructure assets are commonly assumed to be safer than other equity investments. This relative safety is then described as an attractive risk-return ratio, stable cash flow in the longer run and an inherent link to inflation.

Two recent studies present a detailed investigation into the financial characteristics of infrastructure. They compliment earlier work by Newell and Peng (2007), Newell and

Peng (2008), Dechant and Finkenzeller (2009) and Sawant (2010). The first study by Bitsch, Buchner and Kaserer (2010) details the return and risk characteristics of a sample of 363 fully realised infrastructure deals. The second study by Rödel and Rothballer (2012), investigates the inflation hedging capabilities of infrastructure investment.

Bitsch, Buchner and Kaserer (2010) formulate a number of interesting conclusions after having compared 363 infrastructure deals with 11,223 non-infrastructure deals in the context of a private equity type fund. They find no evidence to confirm that the performance of the considered infrastructure deals are correlated with the macroeconomic context in which they are placed. They do find evidence, however, that infrastructure deals provide higher returns on average but have a lower default probability than the non-infrastructure deals in the sample.

When examining long-term cash flows they find no direct evidence that cash flows, measured by cash flows transferred to the fund, are more stable than those provided by non-infrastructure deals. However, when looking at the performance of the deals, the correlation with equity markets in general is high. A possible explanation for this finding is the high gearing of infrastructure deals, thus implying a high dependency on capital markets. While this does not lead to higher default risks, it does imply that the availability of debt funding influences the deal's performance.

In total, the outcome of the analysis differs from what is usually expected. The combination of relatively high returns, lower default risk, gearing levels and correlation with equity markets, suggests that the returns on infrastructure investments are in general of a more nongaussian kind. Given cash flows are in general much less stable and predictable than one might think reinforces this perception. These arguments lead to the conclusion that infrastructure portfolios require dedicated models for optimal portfolio choice.

The second analysis of infrastructure relates to the inflation hedge generally perceived to be embedded in infrastructure investments. Rödel and Rothballer (2012) study a large sample of 824 listed infrastructure companies located in 46 countries and covering a period of 37 years. They conclude that, contrary to a generally held belief, infrastructure does *not* provide a better hedge against inflation than equity in general. Only some infrastructure firms with particularly high pricing power are able to provide a slightly better protection against inflation.

The two studies discussed above and the analyses on which they are based therefore contradict most commonly held beliefs about infrastructure investment. In the next section we will show more convincingly that infrastructure returns are in fact not much different from most other considered risky assets. They show nongaussian returns, which are highly interdependent; additionally their returns vary extensively with economic cycles as borrowing costs fluctuate and demand increases or decreases. What makes infrastructure different is its investment horizon which is often much longer than that of other assets. This implies that dynamic hedging and portfolio choice models based on stochastic models will not add more certainty to the selection of portfolios as their predictive power will not remain valid over longer investment periods.

It is within this context that the introduction of a portfolio model, derived within a single period portfolio optimisation framework may be of significant value, as long as it caters for those characteristics common to most risky asset and infrastructure in particular nongaussianity and interdependence.

5.3 Global infrastructure indexes

Having reviewed the limited work carried out up to the present on the analysis of infrastructure returns, it is both interesting and necessary to provide in depth analysis of the performance of infrastructure returns. The data set to be used consists of sectoral indexes for infrastructure stocks. The choice of test data was in part determined by the availability of data and in part by the requirement to avoid selection bias as well as country effects. We choose the Dow Jones Brookfield Global Infrastructure indexes, introduced in 2003, because they represent a large proportion of the exchange traded infrastructure assets globally.

The indexes are split into eight sectors. The data are separated along eight infrastructure sectors: Airports (DJBAR), Ports (DJBPR), Water (DJBWR), Communication Infrastructure (DJBCM), Oil and Gas Transport and Storage (DJBOS), Electricity Transmission (DJBTD), Toll roads (DJBTR), and Diversified operators (DJBDV). The stocks contained in the indexes are global and have been selected based on certain criteria of size and activity. Table 5-1 presents an overview of our sample.

Table 5-1: Overview of selected infrastructure indexes

	Dow Jones Brookfield Infrastructure Indexes							
Name	Airport Index	Communication Infrastructure	Diversified Infrastructure	Oil Storage and Transport	Port Infrastructure	Toll Roads & Bridges	Electricity Transmission & Distribution	Water Treatment and Distribution
Ticker	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
Type of Data	Market Weighted Stock Index							
Starting Date	01/03/2003							
End Date	01/03/2012							
Frequency	weekly							
Number of Observations	480							

The primary criterion for selection of the data is of course its capacity to represent the financials characteristics of infrastructure assets in general. The descriptive statistics of the sample in Table 5-2, Table 5-3 and Table 5-4 illustrate how the selected data do represent infrastructure assets well. The data are not gaussian as may be seen from the elevated values for kurtosis and skewness and thereafter confirmed by all three normality tests. Moreover, the degree of nongaussianity is not uniform for infrastructure in general and affects some sectors more than others. Specific sectors that are generally thought to have strong dependence on local economic growth, such as electricity and water consumption, tend to show higher levels of nongaussianity than for example communication infrastructure or airports. One explanation might be that, as part of a global network, airports provide a kind of hedge against economic cycles in the home country. And finally the correlation between the sectors is not particularly high; it fluctuates between 0.4 and 0.8, but is sufficiently strong for any portfolio of assets to have a significant dependence on all other sectors.

When we look at the sub-segments of the total sample, supergaussianity seems not to be present in the in-sample segment because kurtosis levels drop significantly. The correlation levels also drop significantly and do not go above 0.515, the correlation between water distribution and electricity transmission which implies that during an economic expansion, infrastructure portfolios assets will behave very differently than during recession periods and importantly that an economic upswing is a poor indicator of the performance of the portfolio during the crisis.

When we examine the "crisis" segment represented in Table 5-4 we can confirm the supergaussian returns, as well as the levels of correlation. And lastly the two-sample Kolmogorov-Smirnov (KS) test indicates large disparity between the distributions of the various infrastructure sectors.

Table 5-2: Summary statistics, correlation matrix and normality tests for the total test sample running from March 2003 to March 2012

Summary Statistics

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
Mean	0.0007	0.0032	0.0016	0.0004	0.0015	0.0010	0.0009	0.0013
Ann. Mean	0.0362	0.1584	0.0824	0.0216	0.0736	0.0508	0.0474	0.0666
Standard Deviation	0.0101	0.0167	0.0105	0.0057	0.0102	0.0076	0.0068	0.0088
Ann. Standard Deviation	0.0712	0.1180	0.0744	0.0404	0.0720	0.0538	0.0483	0.0622
Skewness	0.0191	0.7012	-0.4028	0.0480	0.4192	-0.4717	0.2520	0.4779
Kurtosis	2.2362	2.1591	2.0430	1.3084	2.0937	1.0334	1.5357	2.4098

Correlation Matrix

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
DJBART	1	0.167	0.195	0.202	0.375	0.262	0.344	0.420
p-value		*(0.02)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBCMT		1	0.182	0.398	0.124	0.102	0.169	0.134
p-value			*(0.01)*	*(0.00)*	*(0.07)*	*(0.14)*	*(0.01)*	*(0.05)*
DJBDVT			1	0.351	0.181	0.447	0.244	0.202
p-value				*(0.00)*	*(0.01)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBOST				1	0.294	0.454	0.510	0.323
p-value					*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBPRT					1	0.323	0.284	0.412
p-value						*(0.00)*	*(0.00)*	*(0.00)*
DJBTRT						1	0.327	0.403
p-value							*(0.00)*	*(0.00)*
DJBTDT							1	0.515
p-value								*(0.00)*
DJBWRT								1

Normality tests

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
JB test	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
p-value	*(0.00)*	*(0.00)*	*(0.00)*	*(0.01)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
KS test	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
p-value	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
Lillie test	TRUE	TRUE	TRUE	FALSE	FALSE	TRUE	FALSE	FALSE
p-value	*(0.02)*	*(0.00)*	*(0.02)*	*(0.37)*	*(0.07)*	*(0.03)*	*(0.38)*	*(0.20)*

Two-sample Kolmogorov-Smirnov Test

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
DJBART	FALSE	TRUE	FALSE	TRUE	FALSE	FALSE	FALSE	FALSE
p-value		*(0.00)*	*(0.12)*	*(0.01)*	*(0.64)*	*(0.48)*	*(0.23)*	*(0.87)*
DJBCMT		FALSE	FALSE	TRUE	FALSE	TRUE	TRUE	TRUE
p-value			*(0.23)*	*(0.00)*	*(0.19)*	*(0.00)*	*(0.00)*	*(0.01)*
DJBDVT			FALSE	TRUE	FALSE	FALSE	TRUE	FALSE
p-value				*(0.00)*	*(0.80)*	*(0.07)*	*(0.02)*	*(0.23)*
DJBOST				FALSE	TRUE	TRUE	FALSE	TRUE
p-value					*(0.01)*	*(0.01)*	*(0.34)*	*(0.04)*
DJBPRT					FALSE	FALSE	FALSE	FALSE
p-value						*(0.28)*	*(0.15)*	*(0.72)*
DJBTRT						FALSE	FALSE	FALSE
p-value							*(0.28)*	*(0.48)*
DJBTDT							FALSE	FALSE
p-value								*(0.41)*
DJBWRT								FALSE

True = Reject null *null = Distribution is normal* *Significance level: 5%*

The normality tests are the Kolmogorov-Smirnov test (KS), the Jarque-Bera Test (JB) and the Lillie test (Lillie). The Two-Sample Kolmogorov-Smirnov tests assess the pairwise difference between probability distributions.

Table 5-3: Summary statistics, correlation matrix and normality tests for the in-sample proportion, running from March 2003 to January 2007

Summary Statistics

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
Mean	0.0007	0.0032	0.0016	0.0004	0.0015	0.0010	0.0009	0.0013
Ann. Mean	0.0362	0.1584	0.0824	0.0216	0.0736	0.0508	0.0474	0.0666
Standard Deviation	0.0101	0.0167	0.0105	0.0057	0.0102	0.0076	0.0068	0.0088
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Skewness	0.0191	0.7012	-0.4028	0.0480	0.4192	-0.4717	0.2520	0.4779
Kurtosis	2.2362	2.1591	2.0430	1.3084	2.0937	1.0334	1.5357	2.4098

Correlation Matrix

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
DJBART	1	0.167	0.195	0.202	0.375	0.262	0.344	0.420
p-value		*(0.02)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBCMT		1	0.182	0.398	0.124	0.102	0.169	0.134
p-value			*(0.01)*	*(0.00)*	*(0.07)*	*(0.14)*	*(0.01)*	*(0.05)*
DJBDVT			1	0.351	0.181	0.447	0.244	0.202
p-value				*(0.00)*	*(0.01)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBOST				1	0.294	0.454	0.510	0.323
p-value					*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBPRT					1	0.323	0.284	0.412
p-value						*(0.00)*	*(0.00)*	*(0.00)*
DJBTRT						1	0.327	0.403
p-value							*(0.00)*	*(0.00)*
DJBTDT							1	0.515
p-value								*(0.00)*
DJBWRT								1

Normality tests

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
JB test	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
p-value	*(0.00)*	*(0.00)*	*(0.00)*	*(0.01)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
KS test	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
p-value	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
Lillie test	TRUE	TRUE	TRUE	FALSE	FALSE	TRUE	FALSE	FALSE
p-value	*(0.02)*	*(0.00)*	*(0.02)*	*(0.37)*	*(0.07)*	*(0.03)*	*(0.38)*	*(0.20)*

Two-sample Kolmogorov-Smirnov Test

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
DJBART	FALSE	TRUE	FALSE	TRUE	FALSE	FALSE	FALSE	FALSE
p-value		*(0.00)*	*(0.12)*	*(0.01)*	*(0.64)*	*(0.48)*	*(0.23)*	*(0.87)*
DJBCMT		FALSE	FALSE	TRUE	FALSE	TRUE	TRUE	TRUE
p-value			*(0.23)*	*(0.00)*	*(0.19)*	*(0.00)*	*(0.00)*	*(0.01)*
DJBDVT			FALSE	TRUE	FALSE	FALSE	TRUE	FALSE
p-value				*(0.00)*	*(0.80)*	*(0.07)*	*(0.02)*	*(0.23)*
DJBOST				FALSE	TRUE	TRUE	FALSE	TRUE
p-value					*(0.01)*	*(0.01)*	*(0.34)*	*(0.04)*
DJBPRT					FALSE	FALSE	FALSE	FALSE
p-value						*(0.28)*	*(0.15)*	*(0.72)*
DJBTRT						FALSE	FALSE	FALSE
p-value							*(0.28)*	*(0.48)*
DJBTDT							FALSE	FALSE
p-value								*(0.41)*
DJBWRT								FALSE

True = Reject null *null = Distribution is normal* *Significance level: 5%*

The normality tests are the Kolmogorov-Smirnov test (KS), the Jarque-Bera Test (JB) and the Lillie test (Lillie). The Two-Sample Kolmogorov-Smirnov tests assess the pairwise difference between probability distributions.

Table 5-4: Summary statistics, correlation matrix and normality tests for the out-of-sample proportion, running from January 2007 to March 2012

Summary Statistics

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
Mean	0.0007	0.0009	-0.0003	0.0001	0.0006	-0.0001	-0.0002	-0.0005
Ann. Mean	0.0343	0.0442	-0.0134	0.0065	0.0275	-0.0050	-0.0095	-0.0243
Standard Deviation	0.0160	0.0177	0.0190	0.0133	0.0247	0.0168	0.0120	0.0146
Ann. Standard Deviation	0.1130	0.1251	0.1346	0.0937	0.1749	0.1188	0.0851	0.1033
Skewness	-0.1591	-0.4150	-0.5118	-1.1745	-0.2850	-0.5272	-1.3400	-0.9615
Kurtosis	3.2737	2.2294	4.4725	7.9806	2.7815	2.1766	5.7299	4.6982

Correlation Matrix

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
DJBART	1	0.504	0.799	0.522	0.610	0.858	0.515	0.487
p-value		*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBCMT		1	0.517	0.743	0.304	0.515	0.627	0.617
p-value			*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBDVT			1	0.571	0.574	0.798	0.586	0.545
p-value				*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBOST				1	0.303	0.553	0.822	0.748
p-value					*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
DJBPRT					1	0.603	0.261	0.321
p-value						*(0.00)*	*(0.00)*	*(0.00)*
DJBTRT						1	0.553	0.533
p-value							*(0.00)*	*(0.00)*
DJBTDT							1	0.819
p-value								*(0.00)*
DJBWRT								1

Normality tests

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
JB test	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
p-value	*(0.00)*	*(0.00)*	*(0.00)*	*(0.01)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
KS test	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE	TRUE
p-value	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*	*(0.00)*
Lillie test	TRUE	TRUE	TRUE	FALSE	FALSE	TRUE	FALSE	FALSE
p-value	*(0.02)*	*(0.00)*	*(0.02)*	*(0.37)*	*(0.07)*	*(0.03)*	*(0.38)*	*(0.20)*

Two-sample Kolmogorov-Smirnov Test

	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT
DJBART	FALSE	FALSE	FALSE	TRUE	FALSE	FALSE	TRUE	FALSE
p-value	*1*	*(0.70)*	*(0.89)*	*(0.01)*	*(0.25)*	*(0.70)*	*(0.05)*	*(0.25)*
DJBCMT		FALSE	FALSE	TRUE	FALSE	FALSE	TRUE	FALSE
p-value		*1*	*(0.62)*	*(0.01)*	*(0.41)*	*(0.62)*	*(0.02)*	*(0.11)*
DJBDVT			FALSE	TRUE	FALSE	FALSE	FALSE	FALSE
p-value			*1*	*(0.03)*	*(0.41)*	*(0.97)*	*(0.17)*	*(0.55)*
DJBOST				FALSE	TRUE	TRUE	FALSE	FALSE
p-value				*1*	*(0.00)*	*(0.02)*	*(0.55)*	*(0.09)*
DJBPRT					FALSE	FALSE	TRUE	TRUE
p-value					*1*	*(0.30)*	*(0.01)*	*(0.02)*
DJBTRT						FALSE	FALSE	FALSE
p-value						*1*	*(0.11)*	*(0.41)*
DJBTDT							FALSE	FALSE
p-value							*1*	*(0.48)*
DJBWRT								FALSE

True = Reject null *null = Distribution is normal* *Significance level: 5%*

The normality tests are the Kolmogorov-Smirnov test (KS), the Jarque-Bera Test (JB) and the Lillie test (Lillie). The Two-Sample Kolmogorov-Smirnov tests, assess the pairwise difference between probability distributions.

All the afore mentioned characteristics lead us to the important conclusion that infrastructure is indeed similar to equity in general, for at least part of its sectors and that standard mean-variance methods for portfolio choice therefore cannot lead to portfolios with the desired characteristics.

5.4 Airport operators

The context in which the research for this thesis was conducted had more than one specific focus. The main focus was the development of portfolio optimisation models in the context of classical financial economics, which could extend the present state of the art in the field of higher moment portfolio choice models. The choice to proceed in this direction was in large part motivated by some knowledge of the nature of infrastructure returns.

In this particular context, however, airport returns do represent an interesting case. Contrary to infrastructure in general, their return characteristics differ slightly, as will be shown below. Their returns are nongaussian, but, on a global scale they seem to depend less on each other, as the performance of local airports does not directly impact the performance of a foreign local airport.

Therefore we consider a second dataset consisting of airport operators around the world. The dataset is constructed using Bloomberg's sector classification and from this classification we eliminate all firms that are not exclusively active as airport operators. All airport operators listed later than January 2003, are also kept out of the analysis in order to keep the sample consistent.

Table 5-5: Overview of selected airport operators

Overview of the sample of 19 global airport operators quoted on an exchange between January 2003 and May 2013. Ticker symbols and description of the airports managed are given.

Name	Ticker	Description	Start date	End date	Type
GEMINA SPA	GEM IM Equity	Rome Airport Operator Italy	08-Jan-03	25-May-13	Weekly Closing
JAPAN AIR TERMIN	9706 JP Equity	Tokyo International Airport Japan	08-Jan-03	25-May-13	Weekly Closing
KOREA AIRPORT	005430 KS Equity	Incheon International Airport Korea	08-Jan-03	25-May-13	Weekly Closing
GRUPO AEROPORT-B	ASURB MM Equity	Operator in 13 cities Mexico	08-Jan-03	25-May-13	Weekly Closing
SHANG INTL AIR-A	600009 CH Equity	Taipei International Airport Taiwan	08-Jan-03	25-May-13	Weekly Closing
BEIJING CAP AI-H	694 HK Equity	Beijing International Airport China	08-Jan-03	25-May-13	Weekly Closing
KOBENHAVNS LUFTH	KBHL DC Equity	Kopenhagen International Airport Denmark	08-Jan-03	25-May-13	Weekly Closing
MALAYSIA AIRPORT	MAHB MK Equity	Kuala Lumpur International Airport Malaysia	08-Jan-03	25-May-13	Weekly Closing
SYDNEY AIRPORT	SYD AU Equity	Sydney International Airport Australia	08-Jan-03	25-May-13	Weekly Closing
SHENZ AIRPORT-A	000089 CH Equity	Shenzhen Bao'an International Airport China	08-Jan-03	25-May-13	Weekly Closing
FRAPORT AG	FRA GR Equity	Frankfurt Airport	08-Jan-03	25-May-13	Weekly Closing
XIAMEN INTERNATI	600897 CH Equity	Xiamen International Airport	08-Jan-03	25-May-13	Weekly Closing
AUCKLAND AIRPORT	AIA NZ Equity	Auckland Airport New Zealand	08-Jan-03	25-May-13	Weekly Closing
HAINAN MEILAN-H	357 HK Equity	Hainan Meilan International Airport	08-Jan-03	25-May-13	Weekly Closing
FLUGHAFEN ZU-REG	FHZN SW Equity	Zurich Airport Switzerland	08-Jan-03	25-May-13	Weekly Closing
FLUGHAFEN WIEN	FLU AV Equity	Vienna Airport Austria	08-Jan-03	25-May-13	Weekly Closing
MALTA INTL AIR-A	MIA MV Equity	Malta International Airport	08-Jan-03	25-May-13	Weekly Closing
AEROPORTO DI FIR	AFI IM Equity	Florence Airport Italy	08-Jan-03	25-May-13	Weekly Closing
AERODROM LJUBLJA	AELG SV Equity	Ljubljana Airport Slovenia	08-Jan-03	25-May-13	Weekly Closing

The selected sample now consists of 19 listed airport operators running from 8 January 2003 to 25 May 2013 as shown in Table 5-5. The data are weekly dividend and stock split adjusted closing prices quoted in US Dollars, from which weekly returns are derived. Table 5-6 provides an overview of the summary statistics related to the selected sample and leads to several interesting conclusions.

The index representing the airport sector in the previously studied dataset pointed towards generally subgaussian behaviour, albeit with an above average volatility. The present sample tells a similar story. On average, the airport stocks assume a subgaussian behaviour with low kurtosis values and high volatility. Skewness levels vary around zero, indicating symmetry. Some stocks, however, show either very low or very high levels of kurtosis. In general therefore, the sample looks very similar to the sample of infrastructure indexes, where nongaussian behaviour generated by a large number of different distributions dominated the general picture.

Table 5-6: Summary statistics of selected airport operators

Summary statistics for the total sample of 19 airport operators, computed using weekly returns data adjusted for dividends and stock splits.

Name	Ticker	Mean	Ann. Mean	St. Dev.	Ann. St. Dev.	Skewness	Kurtosis
GEMINA SPA	GEM IM	0.001	0.056	0.000	0.001	0.412	-0.989
JAPAN AIR TERMIN	9706 JP	0.002	0.129	0.049	0.355	0.364	2.734
KOREA AIRPORT	005430 KS	0.004	0.183	0.061	0.442	0.453	2.534
GRUPO AEROPORT-B	ASURB MM	0.005	0.277	0.045	0.325	-0.034	3.522
SHANG INTL AIR-A	600009 CH	0.003	0.147	0.046	0.329	-0.283	4.511
BEIJING CAP AI-H	694 HK	0.004	0.205	0.061	0.436	0.710	6.532
KOBENHAVNS LUFTH	KBHL DC	0.004	0.205	0.039	0.278	1.031	11.363
MALAYSIA AIRPORT	MAHB MK	0.004	0.196	0.038	0.276	0.575	4.423
SYDNEY AIRPORT	SYD AU	0.005	0.238	0.049	0.353	-0.427	2.394
SHENZ AIRPORT-A	000089 CH	0.001	0.071	0.046	0.329	-0.149	1.888
FRAPORT AG	FRA GR	0.003	0.175	0.047	0.338	-0.532	2.959
XIAMEN INTERNATI	600897 CH	0.002	0.130	0.051	0.371	-0.447	4.981
AUCKLAND AIRPORT	AIA NZ	0.003	0.155	0.038	0.277	-0.160	4.654
HAINAN MEILAN-H	357 HK	0.003	0.176	0.066	0.479	1.248	7.624
FLUGHAFEN ZU-REG	FHZN SW	0.006	0.314	0.044	0.319	0.450	3.532
FLUGHAFEN WIEN	FLU AV	0.002	0.119	0.049	0.354	-0.567	7.776
MALTA INTL AIR-A	MIA MV	0.002	0.124	0.031	0.223	0.288	2.469
AEROPORTO DI FIR	AFI IM	0.001	0.064	0.041	0.295	1.047	4.404
AERODROM LJUBLJA	AELG SV	0.001	0.064	0.051	0.370	0.352	4.290

In our aim to carry out further tests, the sample is thus split in a similar way as was done for the index sample. The sub-sample used as in-sample data in future tests runs from 8 January 2003 to 27 December 2006. The out-of-sample data runs from 3 January 2007 to 25 May 2013. As was previously the case, the in-sample data represents a period of economic expansion while the out-of-sample data presents a period of global economic difficulty.

The results are again generally similar to what was observed for the index data, with one crucial difference. When considering the in-sample data, the kurtosis levels are generally lower and below the value of 3, where as for the out-sample data these values tend to increase on average above 3. As expected this indicates that the crisis period provides more extreme returns. The key difference is that there are several outliers in the in-sample data in the present case. These outliers would have been averaged out in the context of an index.

Table 5-7: In-sample summary statistics

Sample running from 8 January 2003 to 27 December 2006. The data are weekly dividend adjusted stock returns for the 19 selected airport operators.

Name	Ticker	Mean	Ann. Mean	St. Dev.	Ann. St. Dev.	Skewness	Kurtosis
GEMINA SPA	GEM IM	0.001	0.067	0.000	0.001	0.186	-1.157
JAPAN AIR TERMIN	9706 JP	0.004	0.190	0.042	0.302	-0.435	1.693
KOREA AIRPORT	005430 KS	0.009	0.475	0.064	0.462	0.787	2.342
GRUPO AEROPORT-B	ASURB MM	0.007	0.383	0.042	0.299	0.623	1.046
SHANG INTL AIR-A	600009 CH	0.006	0.289	0.037	0.265	-1.301	10.550
BEIJING CAP AI-H	694 HK	0.006	0.321	0.044	0.318	-0.059	1.019
KOBENHAVNS LUFTH	KBHL DC	0.008	0.423	0.038	0.274	3.167	23.914
MALAYSIA AIRPORT	MAHB MK	0.003	0.139	0.036	0.261	1.176	4.247
SYDNEY AIRPORT	SYD AU	0.008	0.440	0.038	0.278	0.314	0.974
SHENZ AIRPORT-A	000089 CH	0.003	0.157	0.043	0.308	-0.269	1.570
FRAPORT AG	FRA GR	0.008	0.390	0.039	0.281	0.248	1.765
XIAMEN INTERNATI	600897 CH	0.000	0.019	0.045	0.322	-1.661	10.579
AUCKLAND AIRPORT	AIA NZ	0.004	0.213	0.031	0.223	-0.470	1.454
HAINAN MEILAN-H	357 HK	0.002	0.102	0.053	0.385	0.842	3.572
FLUGHAFEN ZU-REG	FHZN SW	0.012	0.643	0.049	0.356	0.717	4.758
FLUGHAFEN WIEN	FLU AV	0.006	0.301	0.033	0.235	-0.322	1.388
MALTA INTL AIR-A	MIA MV	0.004	0.225	0.030	0.216	0.368	0.963
AEROPORTO DI FIR	AFI IM	0.006	0.291	0.044	0.316	1.590	6.356
AERODROM LJUBLJA	AELG SV	0.007	0.385	0.037	0.268	0.500	0.729

Table 5-8: Out-sample summary statistics

Sample running from 3 January 2007 to 25 May 2013. The data are weekly dividend adjusted stock returns for the 19 selected airport operators.

Name	Ticker	Mean	Ann. Mean	St. Dev.	Ann. St. Dev.	Skewness	Kurtosis
GEMINA SPA	GEM IM	0.001	0.050	0.000	0.001	0.224	-1.137
JAPAN AIR TERMIN	9706 JP	0.002	0.092	0.053	0.385	0.609	2.727
KOREA AIRPORT	005430 KS	0.000	0.004	0.059	0.429	0.175	2.562
GRUPO AEROPORT-B	ASURB MM	0.004	0.212	0.047	0.339	-0.294	4.301
SHANG INTL AIR-A	600009 CH	0.001	0.060	0.050	0.363	0.014	2.978
BEIJING CAP AI-H	694 HK	0.003	0.133	0.069	0.496	0.831	5.967
KOBENHAVNS LUFTH	KBHL DC	0.001	0.071	0.039	0.279	-0.180	3.834
MALAYSIA AIRPORT	MAHB MK	0.004	0.231	0.040	0.286	0.287	4.534
SYDNEY AIRPORT	SYD AU	0.002	0.114	0.054	0.392	-0.509	2.038
SHENZ AIRPORT-A	000089 CH	0.000	0.017	0.047	0.342	-0.082	1.995
FRAPORT AG	FRA GR	0.001	0.042	0.051	0.368	-0.672	2.715
XIAMEN INTERNATI	600897 CH	0.004	0.198	0.055	0.398	-0.075	3.157
AUCKLAND AIRPORT	AIA NZ	0.002	0.119	0.042	0.306	-0.058	4.625
HAINAN MEILAN-H	357 HK	0.004	0.222	0.073	0.528	1.287	7.456
FLUGHAFEN ZU-REG	FHZN SW	0.002	0.112	0.040	0.290	0.006	1.006
FLUGHAFEN WIEN	FLU AV	0.000	0.006	0.057	0.411	-0.477	6.352
MALTA INTL AIR-A	MIA MV	0.001	0.062	0.031	0.227	0.258	3.266
AEROPORTO DI FIR	AFI IM	-0.001	-0.077	0.039	0.279	0.518	1.943
AERODROM LJUBLJA	AELG SV	-0.003	-0.135	0.058	0.419	0.430	3.875

Table 5-9: Total sample correlation matrix and significance levels

The matrix below presents Pearson's correlation coefficients for the complete sample of airport returns running from January 2003 till May 2013. The significance levels are expressed by the p-values for a two-sided t-test with a 5% significance level. All coefficients are significant.

	GEMINA SPA	JAPAN AIR TERMIN	KOREA AIRPORT	GRUPO AEROPORT-B	SHANG INTL AIR-A	BEIJING CAP AI-H	KOBENHAVNS LUFTH	MALAYSIA AIRPORT	SYDNEY AIRPORT	SHENZ AIRPORT-A	FRAPORT AG	XIAMEN INTERNATI	AUCKLAND AIRPORT	HAINAN MEILAN-H	FLUGHAFEN ZU-REG	FLUGHAFEN WIEN	MALTA INTL AIR-A	AEROPORTO DI FIR	AERODROM LJUBLJA
GEMINA SPA	1.000	0.024	0.066	0.021	0.060	0.036	0.072	-0.013	0.069	0.040	0.060	-0.004	0.022	0.011	0.094	0.037	0.042	0.073	0.108
p-value	*0.000*	*0.000*	*0.000*	*0.001*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.017*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
JAPAN AIR TERMIN		1.000	0.132	0.113	0.058	0.169	0.208	0.048	0.170	0.024	0.200	-0.016	0.200	0.133	0.173	0.263	0.060	0.093	0.270
p-value		*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
KOREA AIRPORT			1.000	0.230	0.175	0.322	0.255	0.203	0.226	0.147	0.342	0.027	0.353	0.232	0.127	0.310	0.080	0.171	0.188
p-value			*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
GRUPO AEROPORT-B				1.000	0.144	0.291	0.178	0.241	0.319	0.087	0.354	0.058	0.335	0.199	0.260	0.265	0.168	0.156	0.164
p-value				*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
SHANG INTL AIR-A					1.000	0.281	-0.005	0.182	0.151	0.498	0.155	0.419	0.112	0.211	0.077	0.193	0.075	0.033	0.174
p-value					*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
BEIJING CAP AI-H						1.000	0.212	0.265	0.328	0.125	0.285	0.096	0.343	0.424	0.161	0.393	0.167	0.118	0.280
p-value						*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
KOBENHAVNS LUFTH							1.000	0.148	0.276	0.048	0.347	0.006	0.281	0.169	0.306	0.354	0.229	0.257	0.262
p-value							*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.080*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
MALAYSIA AIRPORT								1.000	0.276	0.128	0.253	0.058	0.172	0.215	0.168	0.166	0.120	0.171	0.169
p-value								*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
SYDNEY AIRPORT									1.000	0.098	0.335	0.045	0.440	0.193	0.314	0.390	0.157	0.220	0.286
p-value									*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
SHENZ AIRPORT-A										1.000	0.149	0.586	0.047	0.101	0.012	0.096	0.029	0.067	0.099
p-value										*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.011*	*0.000*	*0.000*	*0.000*	*0.000*
FRAPORT AG											1.000	0.049	0.403	0.275	0.377	0.489	0.220	0.261	0.297
p-value											*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
XIAMEN INTERNATI												1.000	-0.005	0.075	-0.018	0.036	0.017	0.052	0.070
p-value												*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.002*	*0.000*	*0.000*
AUCKLAND AIRPORT													1.000	0.219	0.302	0.443	0.196	0.249	0.264
p-value													*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
HAINAN MEILAN-H														1.000	0.140	0.236	0.051	0.076	0.184
p-value														*0.000*	*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
FLUGHAFEN ZU-REG															1.000	0.356	0.150	0.223	0.290
p-value															*0.000*	*0.000*	*0.000*	*0.000*	*0.000*
FLUGHAFEN WIEN																1.000	0.248	0.263	0.367
p-value																*0.000*	*0.000*	*0.000*	*0.000*
MALTA INTL AIR-A																	1.000	0.133	0.146
p-value																	*0.000*	*0.000*	*0.000*
AEROPORTO DI FIR																		1.000	0.200
p-value																		*0.000*	*0.000*
AERODROM LJUBLJA																			1.000
p-value																			*0.000*

In a close study of the relationship between the data, we computed the Pearson correlation coefficients for the total sample and these are presented in Table 5-9. The values are relatively low in contrast to the indexes, which is understandable. The indexes represent the general infrastructure sector, which tracks the global stock market in terms of performance. Whereas airport stocks are much more related to the performance of their local economies and therefore do not present high dependency levels.

This conclusion is an important one. Contrary to infrastructure indexes, which are nongaussian but related, the airport stocks yield unrelated nongaussian returns. As a consequence these two samples show two aspects of infrastructure data, both of which a portfolio choice model should be able to handle.

5.5 Conclusion

In the present chapter we have briefly discussed the two data sets on which we will test the portfolio choice method we have derived and elaborated earlier. Using the two selected data sets we aim to provide a comprehensive picture of the performance of our portfolio model while simultaneously understanding the behaviour of infrastructure assets and the sector as a whole.

The two samples are good representatives of the general attributes of infrastructure returns. On the one hand, infrastructure returns are considered to be volatile and nongaussian, with limited proof of increased stability in terms of cash flow. On the other hand infrastructure returns depending on the considered sector, can be either highly dependent or independent across lower and higher moments. Both samples characterise these artefacts.

6

Generalized Portfolio Choice with Independent Components: Applications in Infrastructure Investment

6.1 Introduction

Having described the two samples used in this and the next chapter, we consider the empirical tests and the performance of our portfolio choice model. In order to adequately assess the model presented in chapter 4, we will subject it to two separate tests. First, the entire data sample is applied in the computation of three sets of portfolios for each of the three considered utility functions. The results will be plotted against the efficient frontier, as well as in a mean-skewness (ES) and mean-kurtosis (EK) graph. This format allows us to assess the in-sample performance of the Generalised ICA method when compared with the standard industry benchmark.

Subsequently, we take a proportion of the sample and consider it to be the in-sample proportion as mentioned in the previous chapter. We use this portion to calibrate EV and ICA portfolios for different levels of risk aversion and for the three utility curves in the case of the ICA method. The remaining portion of the sample is considered out-of-sample data. The split is taken at the end of 2006 so that we obtain a de facto crisis sample, starting in 2007 and running to the end of 2011. The first segment between 2003 and 2006 is the in-sample segment. This second test will clarify how the method performs when used in actual portfolio choice.

Plots for the EV, ES and EK space will be made in order to assess the impact of the higher moments on the performance of the portfolio. These experiments will allow us to show how the ICA method can be judged as a more useful tool in the selection of infrastructure portfolios.

6.2 Empirical results

Using the total infrastructure indexes sample running from 2003 to 2012 we calibrate four sets of portfolios. Firstly, a set of twenty EV efficient portfolios is measured along the efficient frontier between the minimum and maximum risk portfolio. Secondly, three sets of portfolios are calibrated, one for each utility curve. For each of the portfolios and for each utility an EV, mean-skewness (ES) and the mean-kurtosis (EK) diagram is plotted. The result is shown in Figure 6-1.

The three utility curves depict very different results. The mean-variance performance of the different portfolios is roughly in line with EV portfolios and the ES and EK diagrams both sets are also remarkably similar to each other. The log and power utility functions have portfolios concentrated around the most risky part of the efficient frontier. Their levels of skewness and kurtosis are, however, also the lowest in the sample.

This conclusion is an important one. In implies that the ICA model is capable of finding the efficient frontier. This is of importance as it implies that the technique is not penalised by the fact additional moments and characteristics are considered in the in-sample tests. It also indicates that traditional portfolio choice mechanisms, such as Sharpe's ratio and the CAPM, remain valid for the choice of a portfolio among the various risk options presented for different risk aversion coefficients.

Figure 6-1: In-sample EV, ES and EK graphs

The nine graphs presented below give an overview of the portfolios constructed using the ICA method and standard EV portfolios using the total sample. The first row of graphs shows the EV graphs for both methods and for three different utility curves. The second row shows Expected return-Skewness (ES) graphs, while the third row shows Expected return-Kurtosis (EK) graphs. A solid line represents the ICA portfolios, EV portfolios are represented by a dashed line.

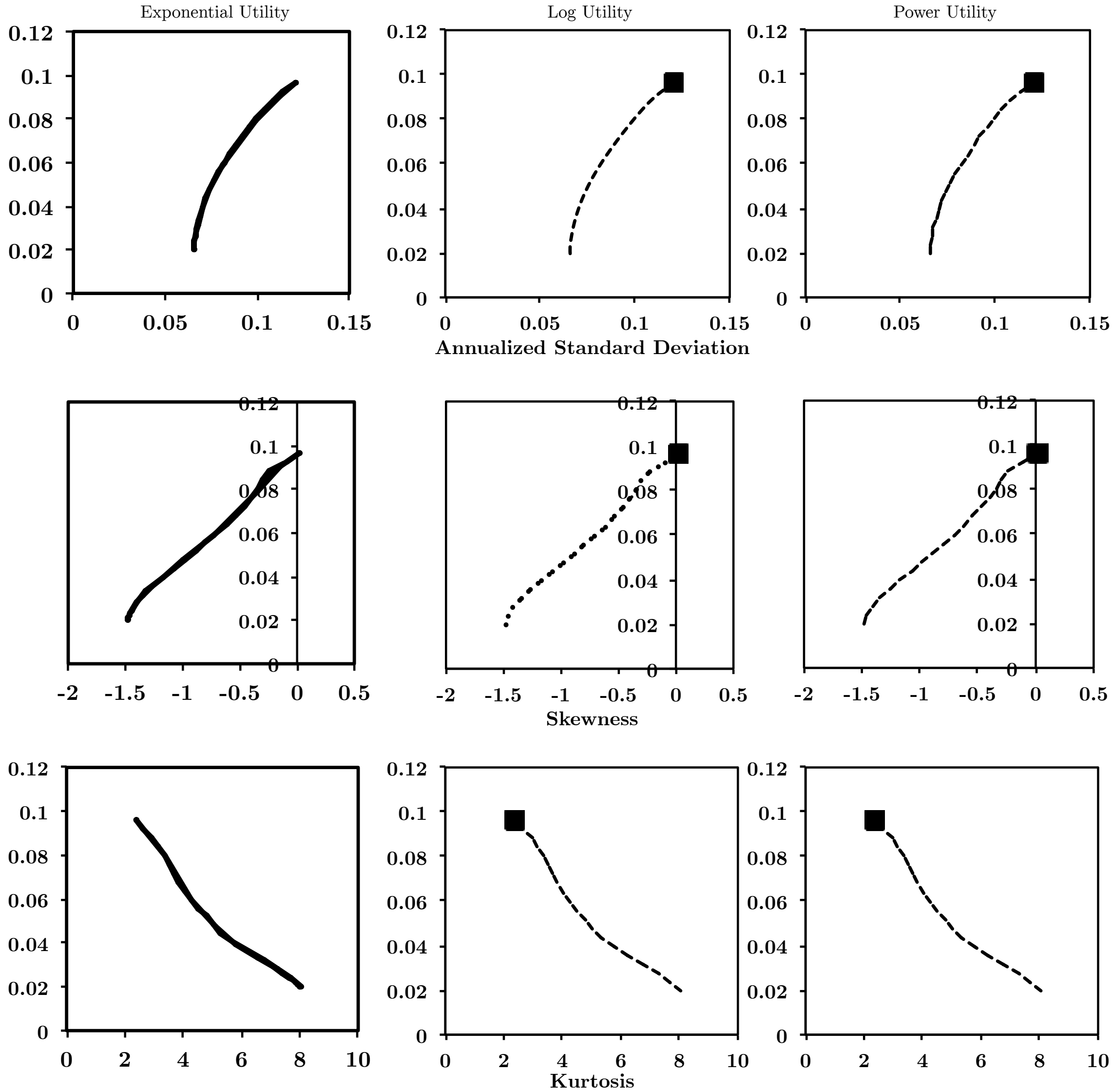


Table 6-1: In-sample portfolio weights and statistics

	ICA portfolio weights									EV portfolio weights								
ϑ	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe
0.5	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.798	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.798
1	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.798	0.00	0.91	0.00	0.00	0.09	0.00	0.00	0.00	0.811
1.5	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.798	0.00	0.82	0.00	0.00	0.18	0.00	0.00	0.00	0.816
2	0.00	0.88	0.00	0.00	0.11	0.00	0.00	0.00	0.813	0.06	0.76	0.00	0.00	0.18	0.00	0.00	0.00	0.812
2.5	0.13	0.69	0.00	0.00	0.17	0.00	0.01	0.00	0.804	0.14	0.69	0.00	0.00	0.17	0.00	0.00	0.00	0.805
3	0.15	0.45	0.00	0.00	0.13	0.00	0.27	0.00	0.728	0.14	0.65	0.00	0.00	0.16	0.00	0.05	0.00	0.795
3.5	0.15	0.31	0.00	0.00	0.11	0.00	0.42	0.00	0.643	0.14	0.60	0.00	0.00	0.16	0.00	0.11	0.00	0.783
4	0.15	0.22	0.00	0.05	0.10	0.00	0.48	0.00	0.561	0.14	0.55	0.00	0.00	0.15	0.00	0.16	0.00	0.769
4.5	0.15	0.15	0.00	0.13	0.09	0.00	0.48	0.00	0.489	0.14	0.50	0.00	0.00	0.14	0.00	0.21	0.00	0.752
5	0.15	0.10	0.00	0.19	0.08	0.00	0.49	0.00	0.437	0.15	0.46	0.00	0.00	0.13	0.00	0.26	0.00	0.731
5.5	0.15	0.07	0.00	0.22	0.07	0.00	0.49	0.00	0.399	0.15	0.41	0.00	0.00	0.13	0.00	0.32	0.00	0.708
6	0.14	0.05	0.00	0.25	0.07	0.00	0.49	0.00	0.371	0.15	0.36	0.00	0.00	0.12	0.00	0.37	0.00	0.680
6.5	0.14	0.03	0.00	0.27	0.07	0.00	0.49	0.00	0.351	0.15	0.32	0.00	0.00	0.11	0.00	0.42	0.00	0.647
7	0.14	0.02	0.00	0.28	0.07	0.00	0.49	0.00	0.335	0.15	0.27	0.00	0.00	0.11	0.00	0.48	0.00	0.610
7.5	0.14	0.01	0.00	0.29	0.06	0.01	0.49	0.00	0.323	0.15	0.22	0.00	0.05	0.10	0.00	0.48	0.00	0.568
8	0.14	0.01	0.00	0.30	0.06	0.01	0.49	0.00	0.314	0.15	0.18	0.00	0.10	0.09	0.00	0.48	0.00	0.521
8.5	0.14	0.00	0.00	0.30	0.06	0.01	0.49	0.00	0.307	0.15	0.13	0.00	0.15	0.08	0.00	0.49	0.00	0.471
9	0.14	0.00	0.00	0.30	0.06	0.01	0.49	0.00	0.306	0.15	0.09	0.00	0.21	0.08	0.00	0.49	0.00	0.418
9.5	0.14	0.00	0.00	0.30	0.06	0.01	0.49	0.00	0.305	0.15	0.04	0.00	0.26	0.07	0.00	0.49	0.00	0.360
10	0.14	0.00	0.00	0.30	0.06	0.01	0.49	0.00	0.304	0.13	0.00	0.00	0.31	0.06	0.02	0.49	0.00	0.300

	ICA portfolio weights using log utility									EV portfolio weights								
ϑ	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe
	0.10	0.02	0.02	0.28	0.06	0.05	0.43	0.04	0.80	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80
										0.00	0.91	0.00	0.00	0.09	0.00	0.00	0.00	0.81
										0.00	0.82	0.00	0.00	0.18	0.00	0.00	0.00	0.82
										0.06	0.76	0.00	0.00	0.18	0.00	0.00	0.00	0.81
										0.14	0.69	0.00	0.00	0.17	0.00	0.00	0.00	0.80
										0.14	0.65	0.00	0.00	0.16	0.00	0.05	0.00	0.79
										0.14	0.60	0.00	0.00	0.16	0.00	0.11	0.00	0.78
										0.14	0.55	0.00	0.00	0.15	0.00	0.16	0.00	0.77
										0.14	0.50	0.00	0.00	0.14	0.00	0.21	0.00	0.75
										0.15	0.46	0.00	0.00	0.13	0.00	0.26	0.00	0.73
										0.15	0.41	0.00	0.00	0.13	0.00	0.32	0.00	0.71
										0.15	0.36	0.00	0.00	0.12	0.00	0.37	0.00	0.68
										0.15	0.32	0.00	0.00	0.11	0.00	0.42	0.00	0.65
										0.15	0.27	0.00	0.00	0.11	0.00	0.48	0.00	0.61
										0.15	0.22	0.00	0.05	0.10	0.00	0.48	0.00	0.57
										0.15	0.18	0.00	0.10	0.09	0.00	0.48	0.00	0.52
										0.15	0.13	0.00	0.15	0.08	0.00	0.49	0.00	0.47
										0.15	0.09	0.00	0.21	0.08	0.00	0.49	0.00	0.42
										0.15	0.04	0.00	0.26	0.07	0.00	0.49	0.00	0.36
										0.13	0.00	0.00	0.31	0.06	0.02	0.49	0.00	0.30

	ICA portfolio weights									EV portfolio weights								
ϑ	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe
0.001	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80
0.05	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.00	0.91	0.00	0.00	0.09	0.00	0.00	0.00	0.81
0.1	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.00	0.82	0.00	0.00	0.18	0.00	0.00	0.00	0.82
0.15	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.06	0.76	0.00	0.00	0.18	0.00	0.00	0.00	0.81
0.2	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.14	0.69	0.00	0.00	0.17	0.00	0.00	0.00	0.80
0.25	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.14	0.65	0.00	0.00	0.16	0.00	0.05	0.00	0.79
0.3	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.14	0.60	0.00	0.00	0.16	0.00	0.11	0.00	0.78
0.35	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.14	0.55	0.00	0.00	0.15	0.00	0.16	0.00	0.77
0.4	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.14	0.50	0.00	0.00	0.14	0.00	0.21	0.00	0.75
0.45	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.15	0.46	0.00	0.00	0.13	0.00	0.26	0.00	0.73
0.5	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.15	0.41	0.00	0.00	0.13	0.00	0.32	0.00	0.71
0.55	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.15	0.36	0.00	0.00	0.12	0.00	0.37	0.00	0.68
0.6	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.15	0.32	0.00	0.00	0.11	0.00	0.42	0.00	0.65
0.65	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.15	0.27	0.00	0.00	0.11	0.00	0.48	0.00	0.61
0.7	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.15	0.22	0.00	0.05	0.10	0.00	0.48	0.00	0.57
0.75	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.15	0.18	0.00	0.10	0.09	0.00	0.48	0.00	0.52
0.8	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.80	0.15	0.13	0.00	0.15	0.08	0.00	0.49	0.00	0.47
0.85	0.00	0.99	0.00	0.00	0.01	0.00	0.00	0.00	0.80	0.15	0.09	0.00	0.21	0.08	0.00	0.49	0.00	0.42
0.9	0.00	0.99	0.00	0.00	0.01	0.00	0.00	0.00	0.80	0.15	0.04	0.00	0.26	0.07	0.00	0.49	0.00	0.36
0.95	0.00	0.99	0.00	0.00	0.01	0.00	0.00	0.00	0.80	0.13	0.00	0.00	0.31	0.06	0.02	0.49	0.00	0.30

[a] The three tables above provide the portfolio weights for the EV and ICA based portfolios using the total data sample as in-sample data. The EV portfolios are part of the efficient set and spread across the efficient frontier between minimum risk and maximum available risk. The ICA based portfolios are calibrated using the risk aversion coefficients or parameter provided in the left columns. The Sharpe ratio for each portfolio is provided on the right of each table.

The out-of-sample performance of the ICA method shows a much different picture. Figure 6-2 presents the EV, ES and EK graphs for the out-of-sample performance of portfolios calibrated using the in-sample data, which runs from 2003 to 2006 inclusive. The ICA based portfolios clearly outperform the EV optimal portfolios in all three graphs for all three utilities.

When comparing in- and out-of-sample performance several observations have to be made. First, the in-sample data used to calibrate the portfolios is much less nongaussian than the out-of-sample data, as general levels of Kurtosis drop toward the value of 3. The decomposition of the assets return space through ICA allows for a complete mapping of the interdependencies between the assets. This in turn implies when the economic regime changes; the portfolios that take these interdependencies into account, are able to deal much better with the changed environment.

More interesting, however, are the investment decisions made by the investor using the in-sample performance of the portfolios as a guideline. Table 6-3 shows the in- and out-of-sample performance ratios for risk-adjusted performance of all three considered utilities. If an investor selects a portfolio based on the in-sample risk adjusted performance, where risk is defined as exposure to all moments higher than the first moment, we can observe the investor will be better off selecting an ICA based portfolio. For example, if we imagine that an investor looks for the highest Sharpe ratio, combined with negative skewness and the least possible kurtosis, the out-of-sample performance of the selected portfolio will be better in every case than the equivalent EV portfolio using the same criteria.

Figure 6-2: Out-of-sample EV, ES and EK graphs

The nine graphs presented below give an overview of the out-of-sample performance (January 2007 to January 2012) of portfolios constructed using the ICA method and standard EV portfolios using the in-sample segment running from January 2003 to December 2006 to calibrate them. The first row of graphs shows the EV graphs for both methods and for three different utility curves. The second row depicts Expected return-Skewness graphs, while the third row shows Expected return-Kurtosis graphs. A solid line represents the ICA portfolios, EV portfolios are represented by a dashed line.

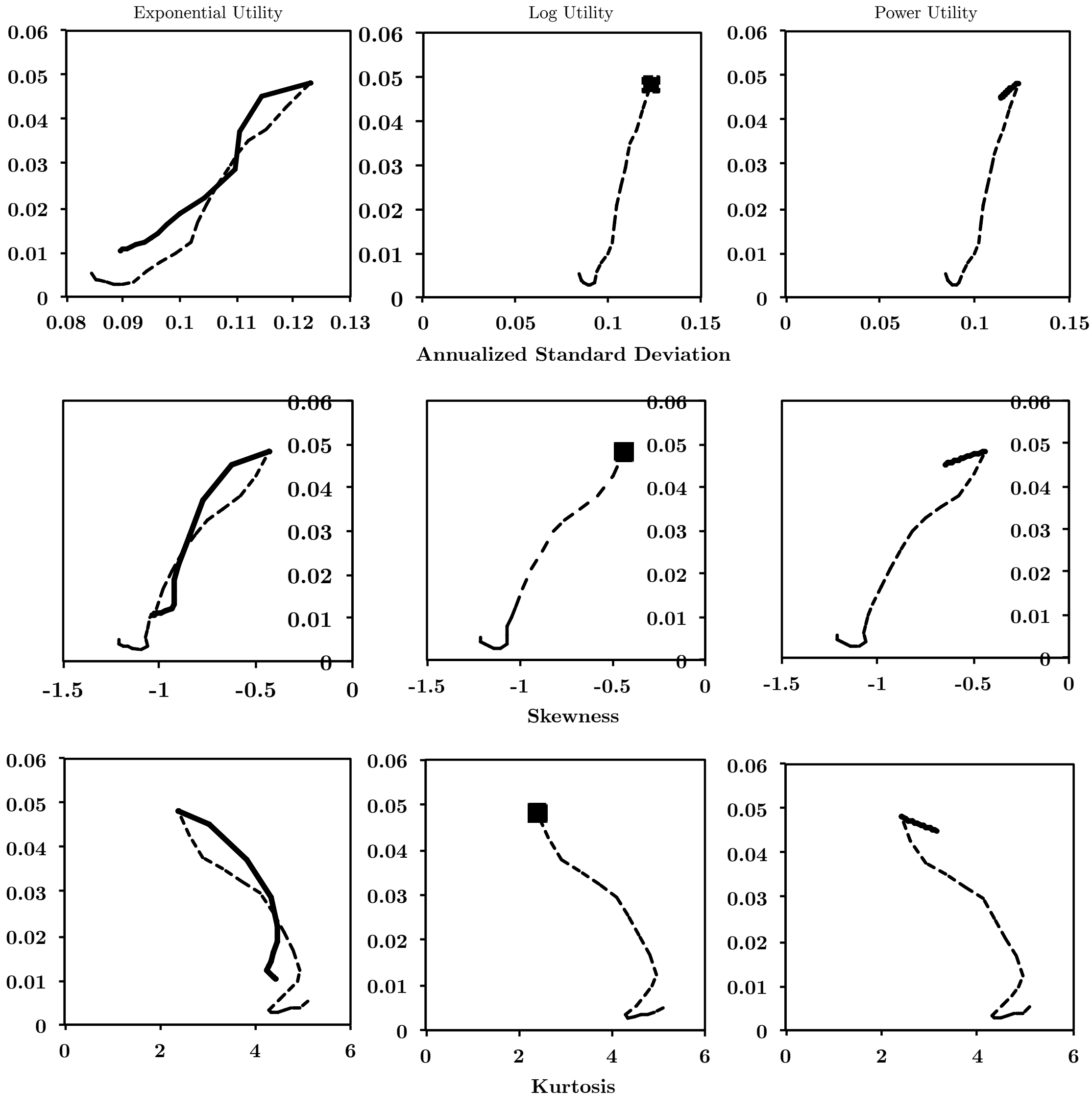


Table 6-2: Portfolio weights using the in-sample data between 2003 and 2006

	ICA portfolio weights for exponential utility									EV portfolio weights								
ϑ	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe
0.5	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.40	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39
1	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.92	0.08	0.00	0.00	0.00	0.00	0.00	0.36
1.5	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.83	0.16	0.00	0.01	0.00	0.00	0.00	0.33
2	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.76	0.18	0.00	0.06	0.00	0.00	0.00	0.31
2.5	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.68	0.21	0.00	0.11	0.00	0.00	0.00	0.29
3	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.60	0.24	0.00	0.16	0.00	0.00	0.00	0.27
3.5	0.00	0.95	0.01	0.00	0.01	0.00	0.00	0.01	0.34	0.00	0.53	0.25	0.00	0.17	0.00	0.00	0.05	0.24
4	0.00	0.93	0.03	0.00	0.02	0.01	0.01	0.01	0.26	0.00	0.46	0.25	0.00	0.18	0.00	0.00	0.11	0.20
4.5	0.01	0.88	0.05	0.00	0.03	0.01	0.01	0.02	0.21	0.00	0.39	0.26	0.00	0.19	0.00	0.00	0.16	0.16
5	0.01	0.80	0.09	0.00	0.05	0.01	0.01	0.02	0.19	0.00	0.31	0.27	0.00	0.20	0.00	0.00	0.22	0.12
5.5	0.01	0.71	0.14	0.01	0.08	0.01	0.01	0.03	0.16	0.00	0.27	0.24	0.00	0.19	0.03	0.07	0.20	0.10
6	0.01	0.62	0.17	0.01	0.11	0.02	0.02	0.05	0.15	0.00	0.23	0.21	0.00	0.18	0.08	0.14	0.17	0.08
6.5	0.01	0.52	0.19	0.01	0.14	0.03	0.02	0.08	0.14	0.00	0.19	0.18	0.00	0.16	0.12	0.21	0.14	0.06
7	0.02	0.41	0.19	0.01	0.16	0.04	0.04	0.13	0.13	0.00	0.15	0.15	0.00	0.15	0.17	0.28	0.11	0.04
7.5	0.03	0.28	0.18	0.02	0.17	0.08	0.09	0.14	0.13	0.00	0.12	0.13	0.06	0.13	0.18	0.29	0.10	0.03
8	0.03	0.17	0.15	0.03	0.16	0.13	0.18	0.15	0.12	0.01	0.09	0.11	0.15	0.12	0.17	0.27	0.09	0.03
8.5	0.06	0.10	0.11	0.16	0.12	0.14	0.18	0.12	0.12	0.03	0.07	0.10	0.23	0.10	0.16	0.25	0.07	0.04
9	0.08	0.04	0.07	0.32	0.09	0.14	0.21	0.06	0.12	0.05	0.04	0.08	0.31	0.08	0.15	0.23	0.06	0.04
9.5	0.08	0.03	0.07	0.34	0.09	0.14	0.21	0.05	0.12	0.07	0.02	0.06	0.39	0.07	0.14	0.21	0.05	0.05
10	0.08	0.02	0.06	0.35	0.08	0.14	0.22	0.05	0.12	0.09	0.00	0.03	0.49	0.05	0.13	0.19	0.03	0.06

	ICA portfolio weights for log utility									EV portfolio weights								
ϑ	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe
	0.10	0.02	0.02	0.28	0.06	0.05	0.43	0.04	0.39	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39
										0.00	0.92	0.08	0.00	0.00	0.00	0.00	0.00	0.36
										0.00	0.83	0.16	0.00	0.01	0.00	0.00	0.00	0.33
										0.00	0.76	0.18	0.00	0.06	0.00	0.00	0.00	0.31
										0.00	0.68	0.21	0.00	0.11	0.00	0.00	0.00	0.29
										0.00	0.60	0.24	0.00	0.16	0.00	0.00	0.00	0.27
										0.00	0.53	0.25	0.00	0.17	0.00	0.00	0.05	0.24
										0.00	0.46	0.25	0.00	0.18	0.00	0.00	0.11	0.20
										0.00	0.39	0.26	0.00	0.19	0.00	0.00	0.16	0.16
										0.00	0.31	0.27	0.00	0.20	0.00	0.00	0.22	0.12
										0.00	0.27	0.24	0.00	0.19	0.03	0.07	0.20	0.10
										0.00	0.23	0.21	0.00	0.18	0.08	0.14	0.17	0.08
										0.00	0.19	0.18	0.00	0.16	0.12	0.21	0.14	0.06
										0.00	0.15	0.15	0.00	0.15	0.17	0.28	0.11	0.04
										0.00	0.12	0.13	0.06	0.13	0.18	0.29	0.10	0.03
										0.01	0.09	0.11	0.15	0.12	0.17	0.27	0.09	0.03
										0.03	0.07	0.10	0.23	0.10	0.16	0.25	0.07	0.04
										0.05	0.04	0.08	0.31	0.08	0.15	0.23	0.06	0.04
										0.07	0.02	0.06	0.39	0.07	0.14	0.21	0.05	0.05
										0.09	0.00	0.03	0.49	0.05	0.13	0.19	0.03	0.06

	ICA portfolio weights for power utility									EV portfolio weights								
ϑ	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe	DJBART	DJBCMT	DJBDVT	DJBOST	DJBPRT	DJBTRT	DJBTDT	DJBWRT	Sharpe
0.001	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39
0.05	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.92	0.08	0.00	0.00	0.00	0.00	0.00	0.36
0.1	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.83	0.16	0.00	0.01	0.00	0.00	0.00	0.33
0.15	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.76	0.18	0.00	0.06	0.00	0.00	0.00	0.31
0.2	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.68	0.21	0.00	0.11	0.00	0.00	0.00	0.29
0.25	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.00	0.60	0.24	0.00	0.16	0.00	0.00	0.00	0.27
0.3	0.00	0.99	0.00	0.00	0.01	0.00	0.00	0.00	0.39	0.00	0.53	0.25	0.00	0.17	0.00	0.00	0.05	0.24
0.35	0.00	0.97	0.00	0.00	0.03	0.00	0.00	0.00	0.40	0.00	0.46	0.25	0.00	0.18	0.00	0.00	0.11	0.20
0.4	0.00	0.95	0.00	0.00	0.05	0.00	0.00	0.00	0.40	0.00	0.39	0.26	0.00	0.19	0.00	0.00	0.16	0.16
0.45	0.00	0.93	0.00	0.00	0.07	0.00	0.00	0.00	0.40	0.00	0.31	0.27	0.00	0.20	0.00	0.00	0.22	0.12
0.5	0.00	0.92	0.00	0.00	0.08	0.00	0.00	0.00	0.40	0.00	0.27	0.24	0.00	0.19	0.03	0.07	0.20	0.10
0.55	0.00	0.90	0.00	0.00	0.10	0.00	0.00	0.00	0.40	0.00	0.23	0.21	0.00	0.18	0.08	0.14	0.17	0.08
0.6	0.00	0.89	0.00	0.00	0.11	0.00	0.00	0.00	0.40	0.00	0.19	0.18	0.00	0.16	0.12	0.21	0.14	0.06
0.65	0.00	0.88	0.00	0.00	0.12	0.00	0.00	0.00	0.40	0.00	0.15	0.15	0.00	0.15	0.17	0.28	0.11	0.04
0.7	0.00	0.86	0.00	0.00	0.14	0.00	0.00	0.00	0.40	0.00	0.12	0.13	0.06	0.13	0.18	0.29	0.10	0.03
0.75	0.00	0.85	0.00	0.00	0.15	0.00	0.00	0.00	0.40	0.01	0.09	0.11	0.15	0.12	0.17	0.27	0.09	0.03
0.8	0.00	0.84	0.00	0.00	0.16	0.00	0.00	0.00	0.40	0.03	0.07	0.10	0.23	0.10	0.16	0.25	0.07	0.04
0.85	0.00	0.83	0.00	0.00	0.17	0.00	0.00	0.00	0.40	0.05	0.04	0.08	0.31	0.08	0.15	0.23	0.06	0.04
0.9	0.00	0.82	0.00	0.00	0.18	0.00	0.00	0.00	0.40	0.07	0.02	0.06	0.39	0.07	0.14	0.21	0.05	0.05
0.95	0.00	0.81	0.00	0.00	0.19	0.00	0.00	0.00	0.39	0.09	0.00	0.03	0.49	0.05	0.13	0.19	0.03	0.06

[a] The three tables above provide the portfolio weights for the EV and ICA based portfolios using the data sample segment running from 2003 till 2006 as in-sample data. The EV portfolios are part of the efficient set and spread across the efficient frontier between minimum risk and maximum available risk. The ICA based portfolios are calibrated using the risk aversion coefficients or parameter provided in the left columns. The Sharpe ratio for each portfolio is provided on the right of each table.

Table 6-3: In-sample and out-of-Sample portfolio risk adjusted performance

	ICA portfolios risk adjusted returns for the exponential utility						EV portfolios risk adjusted returns					
	Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted		Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted	
ϑ	In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
0.5	1.3432	0.3930	0.2259	-0.1116	0.0202	0.0734	1.343	0.393	0.226	-0.112	0.073	0.020
1	1.3431	0.3930	0.2259	-0.1116	0.0202	0.0734	1.390	0.360	0.223	-0.086	0.072	0.016
1.5	1.3431	0.3930	0.2259	-0.1116	0.0202	0.0734	1.439	0.329	0.226	-0.066	0.072	0.013
2	1.3431	0.3930	0.2259	-0.1116	0.0202	0.0734	1.493	0.314	0.236	-0.052	0.073	0.011
2.5	1.3440	0.3929	0.2258	-0.1110	0.0201	0.0733	1.550	0.295	0.255	-0.043	0.075	0.009
3	1.4376	0.3958	0.2326	-0.0721	0.0148	0.0717	1.609	0.272	0.293	-0.036	0.079	0.007
3.5	1.5661	0.3356	0.2568	-0.0477	0.0096	0.0736	1.669	0.238	0.322	-0.029	0.088	0.006
4	1.6707	0.2642	0.3555	-0.0345	0.0067	0.0830	1.730	0.200	0.377	-0.022	0.106	0.005
4.5	1.7428	0.2129	0.5067	-0.0245	0.0050	0.0990	1.784	0.161	0.504	-0.017	0.138	0.003
5	1.7685	0.1851	0.9053	-0.0201	0.0042	0.1209	1.824	0.120	0.909	-0.012	0.190	0.002
5.5	1.7702	0.1646	3.9287	-0.0174	0.0037	0.1419	1.845	0.100	1.635	-0.009	0.203	0.002
6	1.7593	0.1491	-2.1740	-0.0155	0.0033	0.1541	1.853	0.081	-516.320	-0.007	0.179	0.002
6.5	1.7436	0.1372	-0.9778	-0.0141	0.0031	0.1559	1.840	0.060	-1.163	-0.005	0.128	0.001
7	1.7092	0.1298	-0.6222	-0.0130	0.0029	0.1516	1.797	0.038	-0.520	-0.003	0.082	0.001
7.5	1.6505	0.1263	-0.4374	-0.0120	0.0027	0.1423	1.721	0.030	-0.359	-0.002	0.059	0.001
8	1.5966	0.1234	-0.3499	-0.0113	0.0026	0.1303	1.625	0.033	-0.270	-0.003	0.047	0.001
8.5	1.5488	0.1209	-0.3003	-0.0108	0.0025	0.1187	1.512	0.038	-0.209	-0.003	0.038	0.001
9	1.5071	0.1188	-0.2692	-0.0104	0.0024	0.1087	1.376	0.043	-0.165	-0.003	0.031	0.001
9.5	1.4711	0.1171	-0.2482	-0.0102	0.0024	0.1005	1.223	0.047	-0.135	-0.003	0.027	0.001
10	1.4404	0.1157	-0.2335	-0.0099	0.0023	0.0940	1.053	0.060	-0.117	-0.004	0.024	0.001

	ICA portfolios risk adjusted returns for the log utility						EV portfolios risk adjusted returns					
	Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted		Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted	
ϑ	In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
	1.3431	0.3930	0.2259	-0.1117	0.0734	0.0202	1.343	0.393	0.226	-0.112	0.073	0.020
							1.390	0.360	0.223	-0.086	0.072	0.016
							1.439	0.329	0.226	-0.066	0.072	0.013
							1.493	0.314	0.236	-0.052	0.073	0.011
							1.550	0.295	0.255	-0.043	0.075	0.009
							1.609	0.272	0.293	-0.036	0.079	0.007
							1.669	0.238	0.322	-0.029	0.088	0.006
							1.730	0.200	0.377	-0.022	0.106	0.005
							1.784	0.161	0.504	-0.017	0.138	0.003
							1.824	0.120	0.909	-0.012	0.190	0.002
							1.845	0.100	1.635	-0.009	0.203	0.002
							1.853	0.081	-516.320	-0.007	0.179	0.002
							1.840	0.060	-1.163	-0.005	0.128	0.001
							1.797	0.038	-0.520	-0.003	0.082	0.001
							1.721	0.030	-0.359	-0.002	0.059	0.001
							1.625	0.033	-0.270	-0.003	0.047	0.001
							1.512	0.038	-0.209	-0.003	0.038	0.001
							1.376	0.043	-0.165	-0.003	0.031	0.001
							1.223	0.047	-0.135	-0.003	0.027	0.001
							1.053	0.060	-0.117	-0.004	0.024	0.001

	ICA portfolios risk adjusted returns for the power utility						EV portfolios risk adjusted returns					
	Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted		Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted	
ϑ	In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
0.001	1.3483	0.3930	0.2247	-0.1117	0.0730	0.0202	1.343	0.393	0.226	-0.112	0.073	0.020
0.05	1.3433	0.3930	0.2259	-0.1117	0.0734	0.0202	1.390	0.360	0.223	-0.086	0.072	0.016
0.1	1.3432	0.3930	0.2259	-0.1117	0.0734	0.0202	1.439	0.329	0.226	-0.066	0.072	0.013
0.15	1.3432	0.3930	0.2259	-0.1117	0.0734	0.0202	1.493	0.314	0.236	-0.052	0.073	0.011
0.2	1.3432	0.3930	0.2259	-0.1117	0.0734	0.0202	1.550	0.295	0.255	-0.043	0.075	0.009
0.25	1.3432	0.3930	0.2259	-0.1117	0.0734	0.0202	1.609	0.272	0.293	-0.036	0.079	0.007
0.3	1.3431	0.3938	0.2259	-0.1081	0.0734	0.0198	1.669	0.238	0.322	-0.029	0.088	0.006
0.35	1.3431	0.3953	0.2259	-0.1016	0.0734	0.0191	1.730	0.200	0.377	-0.022	0.106	0.005
0.4	1.3431	0.3963	0.2259	-0.0963	0.0734	0.0185	1.784	0.161	0.504	-0.017	0.138	0.003
0.45	1.3431	0.3970	0.2259	-0.0919	0.0734	0.0179	1.824	0.120	0.909	-0.012	0.190	0.002
0.5	1.6707	0.3975	0.3555	-0.0881	0.0830	0.0174	1.845	0.100	1.635	-0.009	0.203	0.002
0.55	1.7593	0.3978	-2.1735	-0.0848	0.1541	0.0170	1.853	0.081	-516.320	-0.007	0.179	0.002
0.6	1.5966	0.3978	-0.3499	-0.0819	0.1303	0.0165	1.840	0.060	-1.163	-0.005	0.128	0.001
0.65	1.4404	0.3977	-0.2335	-0.0794	0.0940	0.0161	1.797	0.038	-0.520	-0.003	0.082	0.001
0.7	1.3562	0.3975	-0.2032	-0.0772	0.0786	0.0157	1.721	0.030	-0.359	-0.002	0.059	0.001
0.75	1.3110	0.3971	-0.1911	-0.0753	0.0718	0.0154	1.625	0.033	-0.270	-0.003	0.047	0.001
0.8	1.2860	0.3967	-0.1854	-0.0735	0.0685	0.0151	1.512	0.038	-0.209	-0.003	0.038	0.001
0.85	1.2720	0.3961	-0.1824	-0.0719	0.0667	0.0148	1.376	0.043	-0.165	-0.003	0.031	0.001
0.9	1.2642	0.3955	-0.1808	-0.0705	0.0657	0.0145	1.223	0.047	-0.135	-0.003	0.027	0.001
0.95	1.2740	0.3947	-0.1828	-0.0692	0.0669	0.0143	1.053	0.060	-0.117	-0.004	0.024	0.001

[a] The three tables above provide risk adjusted performance measures for variance, skewness and kurtosis. For each utility function the set of risk measures is provided for both the EV and the ICA portfolios. The results are based on the 2003-2006 sub-segment treated are as in-sample data leaving the rest of the sample as out-sample data.

The conclusions drawn from Table 6-3 are significant because they indicate that not only does the ICA method show greater resilience to changes in higher moments and therefore greater crisis robustness, but it also allows the investor to make practical use of this capacity by extending the choice of portfolios beyond Sharpe's ratio. The inclusion of higher moments is of importance when managing risk, as shown in Table 6-3.

Several observations should be made at this point. First and foremost, the out of sample tests indicate that the technique does indeed outperform EV models when basing the choice of a portfolio on the same in-sample tests and calibrations with the same portfolio selection tools. This is important as it implies that the selection and literature regarding such decisions remains applicable in practice.

The second important conclusion is the importance gained by the higher moments. As can be seen from the two graphs and tables, the out-of-sample performance of the different portfolios for the different utility curves is clearly more robust for the ICA portfolios when higher moments are considered. This is in fact the end goal of the the derivation of the model, as such outperformance implies robustness to crises and economic cycles.

From the two sets of tests above we can assert that the presented ICA based method has clear advantages over the EV method when long-term infrastructure investments are under scrutiny. This advantage is not limited to a specific investor or utility curve but seems generally to be present in the various tested utilities.

6.3 Conclusion

In the current chapter we have tested the portfolio choice model introduced in chapter 6, a model specifically developed for portfolios of primarily nongaussian assets, which are held in the portfolio for longer periods of time without rebalancing. Infrastructure portfolios are a prominent case of such investments. The model is based on the expected utility maximising framework for single period investments and tested for most commonly used utility functions.

From a financial theoretic point of view, the method approaches the portfolio selection problem in a more comprehensive way than classical modern portfolio theory. Rather than to assume probability beliefs on the part of the investor, the proposed model infers these probability beliefs for historical probabilities of the considered assets and their interdependencies. For an investor with a certain risk aversion profile expressed by a utility curve, our method is able to find the portfolio that best fits that investor's preferences without requiring subjective views on the future from the considered investor.

The results highlight the advantages of using this model. When tested in-sample, the method performs similar to the standard EV method. We turn to the out-of-sample performance; the presented method outperforms the standard EV portfolios. Let us keep in mind that, in this case is that investment decisions based on the in-sample performance of the portfolio lead to better out-of-sample performance than for the EV portfolios. This implies that the derived method provides a possible hedging against shocks to the state-variables, which would trigger a rebalancing of the portfolio in most cases but which the nature of the investment in the present case usually renders impossible.

The model is also easily applicable from a practical point of view, as it does not require ex-ante hypotheses regarding the return distributions of the asset or the utility curve. It is therefore the first model to solve the utility maximising framework analytically for all higher moments and all utility curves while avoiding restrictive assumptions. The model therefore allows for the arrival to closed-form solutions for the multiple integrals, which compose the expected utility maximisation framework.

7

Portfolio Choice with Independent Components: Studying the Case of Airport Operators

7.1 Introduction

Airport operators are an interesting case for infrastructure portfolios. They are usually grouped in single asset type portfolios or groups and perform in a fairly independent manner globally, with only moderate correlation. No particular airport or airport group was the prime focus of the study, therefore it is interesting to consider the sub-sector as a whole in the context of a second set of tests of the portfolio choice model. This chapter is dedicated to analyse the airport sub-sector.

In chapter 5 we showed how the portfolio choice model could be of value in the construction of infrastructure portfolios. The results indicated that the out-of-sample performance of the portfolio choice model, based on Independent Component Analysis will outperform standard EV portfolios in the context of an infrastructure portfolio. These tests alone do not prove the validity of concept in general, but they give a good indication that the model produces robust results.

We will now repeat the tests performed in the previous chapter on the second data set considered in this thesis consisting of 19 airport operators traded on the stock exchange between 2003 and 2013. This additional test is particularly interesting with regard to this thesis for two reasons. Airport stocks, even if they are also part of infrastructure in general, are a special case of infrastructure and permit a robustness testing of the derived method. Secondly, as was seen from the initial statistical analysis, airports seem to have fewer correlated returns than infrastructure indexes in general, which is of interest to our method.

Using the first part of the sample running to 3 January 2007 as in-sample data for the calibration of the model, we consider the performance of the selected portfolios on the remaining sub-sample. It is noteworthy that once again the remaining sub-sample covers a recession period globally, and this fact is indeed visible in the summary statistics presented in chapter 6.

7.2 Empirical results

Similar to chapter 5, we first consider the performance of the ICA based portfolio choice method compared to EV portfolios. The results for portfolios with varying risk aversion coefficients are made available in Figure 7-1.

Broadly speaking, the results look very similar to those presented in chapter 6. The ICA-based portfolios perform close to, or as good as, the EV portfolios. Any performance differences when considering the mean-variance graphs are in favour of the EV portfolios and can be justified through the fact that the in-sample optimisation using two criteria will on the whole be more accurate than the in-sample optimisation using numerous criteria.

The logarithmic utility function is the only one underperforming very clearly. No direct explanation can be given in this case other than the fact that the numerically conducted optimisation might have found a local rather than a global optimum.

When we examine at the ES graphs, the exponential utility seems to underperform the EV portfolios when risk aversion rises. This leads to portfolios that are less negatively skewed than the EV equivalents because of a higher exposure to Copenhagen Airport, which is a stock showing elevated levels of kurtosis and skewness in its in-sample performance. However, when we compare in-sample variance adjusted returns with the equivalent skewness adjusted returns, as shown in Table 7-1, we observe clearly that the equivalent ICA choice would have led to better performance than the EV portfolios. This conclusion is important as it signals that, even if the graphs trace a similar path, and represent a similar efficient set, the strategic implications are more in favour of the ICA portfolios.

Figure 7-1: In-sample portfolio performance

The three times three graphs below depict the in-sample performance of the ICA based method as well as the EV portfolios, using the sub-sample running from 8 January 2003 to 27 December 2006. The portfolios have risk aversion coefficients varying between 0.5 and 10 for the exponential utility and between 0.01 and 1 for the power utility. A solid line represents the ICA portfolios; a dashed line represents the EV portfolios.

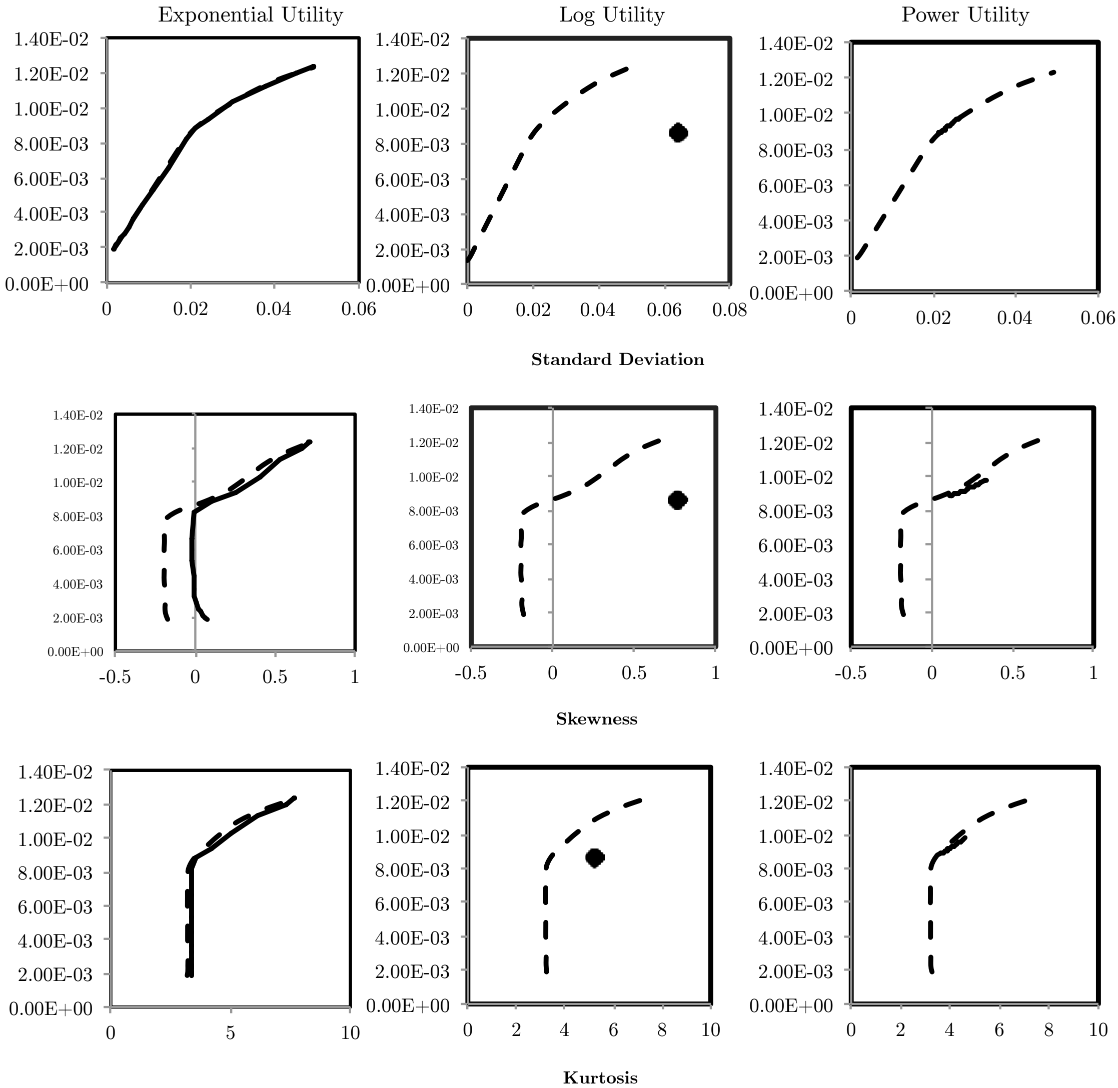


Figure 7-2 depicts the results for the simulation in a similar set of graphs as shown. Three main conclusions can be reached from these graphs.

First and foremost the results and conclusions attained from the study of infrastructure indexes in general remains valid. In all but the case of the logarithmic utility function the performance of the ICA based portfolios is better than that of EV based portfolios. The difference is most visible for the power utility function in the current case, which provides obvious higher returns for each of the risk measures.

When we look closely at the results, we notice that the airport portfolios follow the same structure as the portfolios of infrastructure indexes, in the sense that they cap both skewness and kurtosis in the case of the exponential utility. A similar but opposite argument may be made for the power utility, where the ICA-based portfolios do not have such a cap. In both cases the nature of the utility curve explains the performance figures.

Perhaps our most important finding relates to the differences we can point to with respect to the previous tests. The sole and main difference is the degree to which the ICA portfolios outperform the EV portfolios, which is less large than in earlier tests. The explanation resides with the level of correlation or relation in general between the airport stocks. ICA portfolios offer an additional tool to structure portfolios, which are less dependent on the same fundamental drivers. In our case the stocks are relatively uncorrelated and this finding weakens the advantage of the ICA portfolios in that respect.

Figure 7-2: Out-of-sample performance

The graphs below present the out-of-sample performance of the three considered utility functions for risk aversion coefficients varying between 0.5 and 10 for the exponential utility and 0.01 and 1 for the power utility. A solid line represents the ICA portfolios; a dashed line represents the EV portfolios.

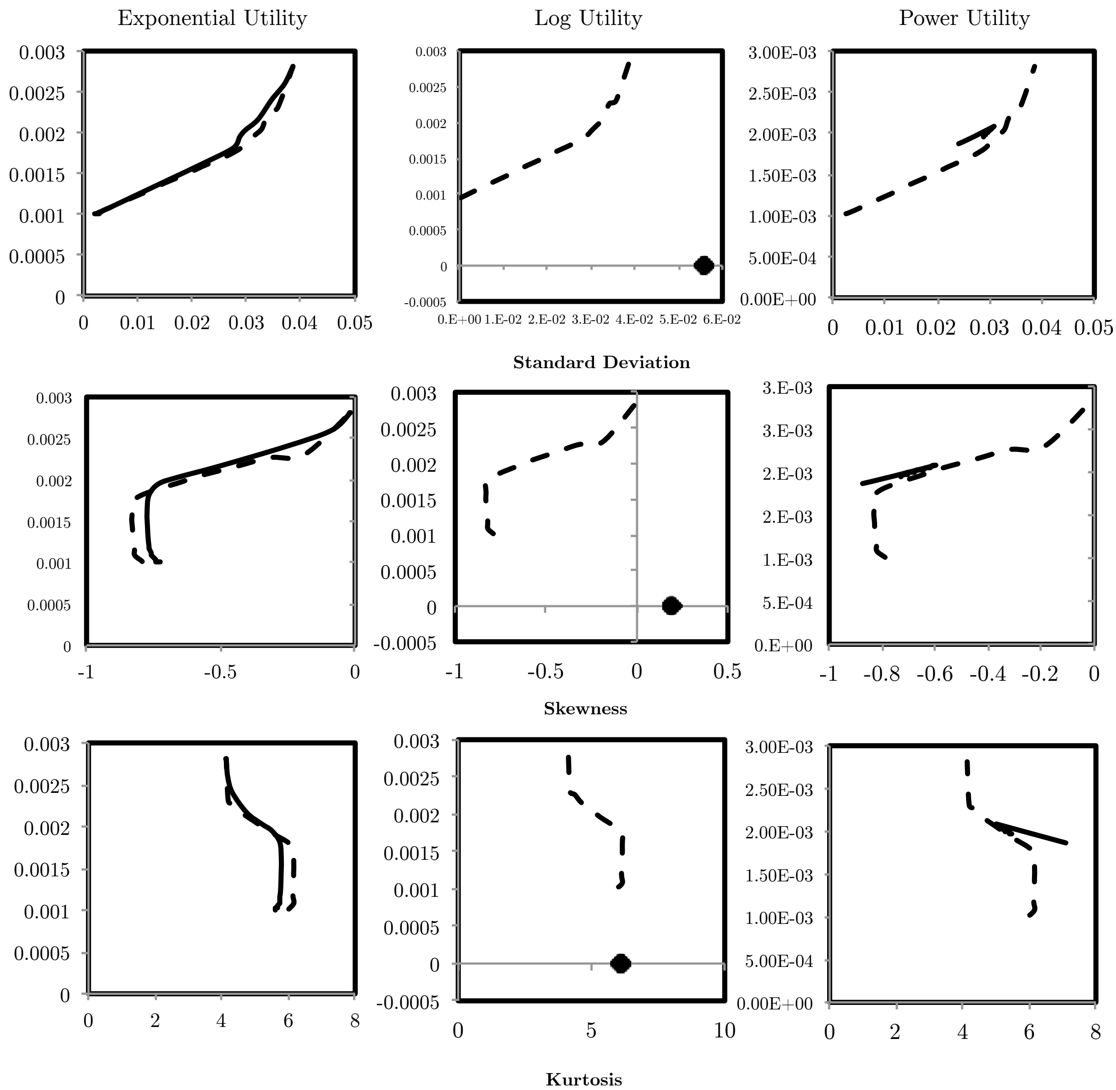


Table 7-1: Risk adjusted performance measures

The performance measures below are risk-adjusted levels of the average returns obtained from the out-of-sample tests. The risk adjustment is performed using one of the three considered moments, the standard deviation, the skewness and the kurtosis.

Exp Utility	ICA portfolios risk adjusted returns						EV portfolio weights					
	Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted		Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted	
ϑ	In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
0.5	1.1581	0.4496	0.0233	-0.0014	0.0006	0.0002	1.1819	0.4146	-0.0104	-0.0013	0.0006	0.0002
1	1.0268	0.3770	0.0332	-0.0014	0.0006	0.0002	1.1819	0.4146	-0.0104	-0.0013	0.0006	0.0002
1.5	0.9148	0.3161	0.0499	-0.0014	0.0006	0.0002	0.7811	0.2221	-0.0129	-0.0013	0.0008	0.0002
2	0.8190	0.2649	0.0813	-0.0014	0.0007	0.0002	0.6441	0.1572	-0.0163	-0.0014	0.0009	0.0002
2.5	0.7373	0.2217	0.1529	-0.0014	0.0008	0.0002	0.5766	0.1257	-0.0191	-0.0015	0.0011	0.0002
3	0.6678	0.1853	0.4223	-0.0015	0.0008	0.0002	0.5356	0.1061	-0.0214	-0.0016	0.0013	0.0002
3.5	0.6088	0.1547	-2.0657	-0.0015	0.0010	0.0002	0.5084	0.0937	-0.0245	-0.0017	0.0015	0.0002
4	0.5590	0.1290	-0.4422	-0.0016	0.0011	0.0002	0.4888	0.0845	-0.0280	-0.0017	0.0017	0.0002
4.5	0.5172	0.1076	-0.3120	-0.0017	0.0013	0.0002	0.4742	0.0774	-0.0303	-0.0018	0.0019	0.0002
5	0.4825	0.0899	-0.2841	-0.0019	0.0016	0.0002	0.4629	0.0724	-0.0337	-0.0019	0.0020	0.0003
5.5	0.4539	0.0755	-0.2950	-0.0021	0.0020	0.0003	0.4538	0.0679	-0.0363	-0.0020	0.0022	0.0003
6	0.4327	0.0658	-1.1239	-0.0024	0.0025	0.0003	0.4463	0.0644	-0.0390	-0.0021	0.0024	0.0003
6.5	0.4178	0.0678	0.0878	-0.0027	0.0025	0.0004	0.4383	0.0625	-0.0719	-0.0023	0.0025	0.0003
7	0.3893	0.0682	0.0369	-0.0031	0.0022	0.0004	0.4201	0.0626	0.1686	-0.0026	0.0025	0.0003
7.5	0.3435	0.0673	0.0251	-0.0043	0.0020	0.0005	0.3926	0.0625	0.0515	-0.0030	0.0024	0.0004
8	0.3002	0.0694	0.0211	-0.0095	0.0018	0.0006	0.3634	0.0624	0.0366	-0.0035	0.0023	0.0004
8.5	0.2655	0.0702	0.0180	-0.0298	0.0017	0.0006	0.3357	0.0646	0.0298	-0.0049	0.0022	0.0005
9	0.2505	0.0729	0.0172	-0.1599	0.0016	0.0007	0.3100	0.0661	0.0255	-0.0071	0.0020	0.0005
9.5	0.2505	0.0730	0.0172	-0.1598	0.0016	0.0007	0.2832	0.0645	0.0214	-0.0122	0.0018	0.0005
10	0.2506	0.0730	0.0172	-0.1547	0.0016	0.0007	0.2504	0.0729	0.0172	-0.1675	0.0016	0.0007

Log Utility	ICA portfolio weights						EV portfolio weights					
	Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted		Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted	
ϑ	In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
	0.1340	0.0000	0.0112	0.0000	0.0017	0.0000	1.1819	0.4146	-0.0104	-0.0013	0.0006	0.0002
							1.1819	0.4146	-0.0104	-0.0013	0.0006	0.0002
							0.7811	0.2221	-0.0129	-0.0013	0.0008	0.0002
							0.6441	0.1572	-0.0163	-0.0014	0.0009	0.0002
							0.5766	0.1257	-0.0191	-0.0015	0.0011	0.0002
							0.5356	0.1061	-0.0214	-0.0016	0.0013	0.0002
							0.5084	0.0937	-0.0245	-0.0017	0.0015	0.0002
							0.4888	0.0845	-0.0280	-0.0017	0.0017	0.0002
							0.4742	0.0774	-0.0303	-0.0018	0.0019	0.0002
							0.4629	0.0724	-0.0337	-0.0019	0.0020	0.0003
							0.4538	0.0679	-0.0363	-0.0020	0.0022	0.0003
							0.4463	0.0644	-0.0390	-0.0021	0.0024	0.0003
							0.4383	0.0625	-0.0719	-0.0023	0.0025	0.0003
							0.4201	0.0626	0.1686	-0.0026	0.0025	0.0003
							0.3926	0.0625	0.0515	-0.0030	0.0024	0.0004
							0.3634	0.0624	0.0366	-0.0035	0.0023	0.0004
							0.3357	0.0646	0.0298	-0.0049	0.0022	0.0005
							0.3100	0.0661	0.0255	-0.0071	0.0020	0.0005
							0.2832	0.0645	0.0214	-0.0122	0.0018	0.0005
							0.2504	0.0729	0.0172	-0.1675	0.0016	0.0007

Power Utility	ICA portfolio weights						EV portfolio weights					
	Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted		Variance Adjusted		Skewness Adjusted		Kurtosis Adjusted	
ϑ	In	Out	In	Out	In	Out	In	Out	In	Out	In	Out
0.001	0.3546	0.0771	-0.0113	-0.0021	0.0016	0.0003	1.1819	0.4146	-0.0104	-0.0013	0.0006	0.0002
0.05	0.3671	0.0677	0.0291	-0.0035	0.0021	0.0004	1.1819	0.4146	-0.0104	-0.0013	0.0006	0.0002
0.1	0.3720	0.0678	0.0303	-0.0034	0.0022	0.0004	0.7811	0.2221	-0.0129	-0.0013	0.0008	0.0002
0.15	0.3763	0.0679	0.0315	-0.0033	0.0022	0.0004	0.6441	0.1572	-0.0163	-0.0014	0.0009	0.0002
0.2	0.3803	0.0680	0.0328	-0.0032	0.0022	0.0004	0.5766	0.1257	-0.0191	-0.0015	0.0011	0.0002
0.25	0.3840	0.0681	0.0343	-0.0032	0.0022	0.0004	0.5356	0.1061	-0.0214	-0.0016	0.0013	0.0002
0.3	0.3874	0.0682	0.0358	-0.0031	0.0022	0.0004	0.5084	0.0937	-0.0245	-0.0017	0.0015	0.0002
0.35	0.3905	0.0682	0.0375	-0.0031	0.0023	0.0004	0.4888	0.0845	-0.0280	-0.0017	0.0017	0.0002
0.4	0.3934	0.0682	0.0393	-0.0030	0.0023	0.0004	0.4742	0.0774	-0.0303	-0.0018	0.0019	0.0002
0.45	0.3961	0.0683	0.0413	-0.0030	0.0023	0.0004	0.4629	0.0724	-0.0337	-0.0019	0.0020	0.0003
0.5	0.3987	0.0683	0.0434	-0.0029	0.0023	0.0004	0.4538	0.0679	-0.0363	-0.0020	0.0022	0.0003
0.55	0.4010	0.0683	0.0457	-0.0029	0.0023	0.0004	0.4463	0.0644	-0.0390	-0.0021	0.0024	0.0003
0.6	0.4033	0.0682	0.0483	-0.0029	0.0024	0.0004	0.4383	0.0625	-0.0719	-0.0023	0.0025	0.0003
0.65	0.4054	0.0682	0.0512	-0.0028	0.0024	0.0004	0.4201	0.0626	0.1686	-0.0026	0.0025	0.0003
0.7	0.4074	0.0682	0.0544	-0.0028	0.0024	0.0004	0.3926	0.0625	0.0515	-0.0030	0.0024	0.0004
0.75	0.4093	0.0682	0.0580	-0.0028	0.0024	0.0004	0.3634	0.0624	0.0366	-0.0035	0.0023	0.0004
0.8	0.4111	0.0681	0.0621	-0.0028	0.0024	0.0004	0.3357	0.0646	0.0298	-0.0049	0.0022	0.0005
0.85	0.4128	0.0681	0.0667	-0.0027	0.0024	0.0004	0.3100	0.0661	0.0255	-0.0071	0.0020	0.0005
0.9	0.4144	0.0681	0.0721	-0.0027	0.0025	0.0004	0.2832	0.0645	0.0214	-0.0122	0.0018	0.0005
0.95	0.4159	0.0680	0.0784	-0.0027	0.0025	0.0004	0.2504	0.0729	0.0172	-0.1675	0.0016	0.0007

Let us now examine the performance of each of the portfolios in detail in Table 7-1. The table represents in- and out-of-sample measures of risk-adjusted returns. The risk adjustment is variable and depends on the chosen risk measure. The three risk measures under scrutiny are the standard deviation, the skewness and the kurtosis.

In Table 7-1, the results indicate that the graphs shown are indeed confirmed by the numbers. When we account for the like-for-like portfolios between the ICA and the EV choice, with regard to variance, skewness or kurtosis adjusted performance measure, our basing an investment decision on either one will lead to better out-of-sample performance using the ICA portfolios.

The impact on investment decisions in airports yields two noteworthy conclusions. First, airports are very similar to other infrastructure assets in terms of their overall performance. Given that they are less correlated globally, leads to a somewhat weaker performance of the portfolios choice system but does not undermine the proposed method.

However, when we consider airport operators in the same country or region, it is apparent from the sample that correlation levels are much higher. And this matter is crucial to airport funds or groups whose regional coverage is usually limited as can be seen from the BAA group of the Manchester Airport Group (MIG). Both operate regionally and own airports that do have correlated activities.

Second, when airport portfolios, funds or groups are considered, the tests have clearly proven that it is very much in the interest of the investor to structure the fund using the ICA method rather than the EV method. The use of ICA will make the portfolio more robust against market and stock market moves; and not only bring but also generate higher returns.

And third, one of the aims of the method was to provide a portfolio choice model that allows for longer-term investments. All tests were performed using longer-term samples, with investments over terms of five years or more. These are investment periods generally considered by investment funds and therefore provide the funds with the necessary tools to make longer-term investment decisions.

7.3 Conclusion

We have repeated the tests performed in chapter 6 using a distinct data set composed of airport operators quoted on the stock market globally. The aim was to test whether the good results obtained in the previous chapter could be replicated for a particular sub-sector of the infrastructure sphere.

Airports are assets that share the characteristics of infrastructure in general, but they also have their own particular attributes. The shared characteristics include the nongaussian aspect and the prominence of higher moments in the asset returns. A particularity is that airport operators around the world are less correlated to each other in terms of their performance than infrastructure assets.

Applying the ICA portfolio choice method to airport assets leads to very similar results to those reported for infrastructure. The lack of correlation impacts on the results as the ICA method benefits from strongly correlated samples. Within the same country or region however, airport operators do tend to show higher correlation levels, as can be seen from our sample. The fact of high correlation levels is of high importance to this set of tests, as airport groups would benefit from a tool, which allows them to structure financially astute investment groups.

The tests performed in this chapter confirm our earlier results. At least for the considered data samples, the results confirm that the concept proposed for the portfolio choice model, provides an interesting starting point for further research.

Summary and Conclusion

One of the principal questions in theoretical and practical financial economics is the optimal allocation of capital assets to portfolios. Markowitz (1952) has provided us with the foundations of the current theory – mean-variance optimisation - while highlighting the theory's possible weaknesses; parameter uncertainty related to the inputs, and the possibility that mean-variance efficient portfolios are not optimal as higher order moments cannot be ignored.

In this thesis we have presented a possible answer to the search for an optimal portfolio choice model, which addresses the highlighted weaknesses of mean-variance optimisation. Building on recent literature in the field of higher moment asset allocation we have proposed an alternative model to address both the need for the inclusion of higher moments and the limitation of estimation risk related to input parameters. We begin from a vantage point very similar to Jondeau and Rockinger (2006) or Jurczenko and Maillet (2006). The main aim, as with the two papers just cited, has been to develop an analytical higher moment solution to the asset allocation problem in the context of the determination of the opportunity set, when the considered capital assets have a behaviour governed by their higher moments.

Through the introduction of Independent Component Analysis, a largely unknown technique in the financial world, we have proposed a fully analytical solution to the expected utility maximising framework and as such, have found a portfolio choice model which includes all higher moments of the considered assets' return distributions. The fact that ICA has been applied successfully in finance in the past has played an important role in the decision to apply this technique in the context of portfolio choice.

ICA allows for a cross-sectional decomposition of the asset space. While such decomposition does not provide any benefits in terms of dimensionality reduction, it nevertheless allows for the complete factorisation of the optimisation problem and a significant simplification of the optimisation. The linearity of the base formulation of ICA is instrumental in our studies. Subsequent use of the properties of the independent components has made it possible to obtain a fully analytical solution to the optimisation problem.

In the thesis we have presented three models in total as well as two sets of empirical tests. First, we derived a straight optimisation of the expected utility maximisation model for CARA investors. The model yielded a number of interesting features and showed how ICA can be applied successfully in the context of portfolio choice. Second we generalised the framework to all von Neumann and Morgenstern utility functions and presented a portfolio choice model that allows for higher moment portfolio selection while reducing the effects of parameter uncertainty. Lastly we showed how diversification can be reformulated in terms of independent components to be fully integrated with the derived portfolio choice model.

We have applied the models to two sets of empirical data. The data consist of the returns of infrastructure indexes and airport operators. Contrary to popular belief, infrastructure does necessarily present a safe heaven for investors. In reality, infrastructure returns are highly nongaussian in a similar fashion as other standard financial returns are. Additionally, infrastructure investment is usually long term, thus requiring a portfolio choice model which accommodates both requirements.

The results of the application have proved to be robust. The optimal portfolios constructed using our methods clearly outperform the standard mean-variance portfolios when out-of-sample returns are considered. This result holds for both the sample of infrastructure indexes, and the sample of airport operators, and for the three tested utility functions.

The concept for a portfolio choice model developed in the present thesis opens the door for two further developments. The concept needs more testing in order to be fully validated as a single period higher moment portfolio choice model. Several interesting avenues for further research are also now available. This additional research can be split approximately into three main areas.

First of all, the application of ICA in finance should continue to be investigated. Several papers which have been discussed in this thesis, have shown ICA to be a valid framework in the analysis of financial data, but a more structured and detailed study of the added benefit beyond the mathematical, and the interpretability of the independent components, could be of significant value.

Secondly, Independent Component Analysis has several possible extensions, a number of which have been mentioned in the earlier chapters. These extensions are meant to bring the method closer to the true nature of the data upon which it is applied. Non-linear ICA, as well as noisy ICA are examples that could potentially increase the accuracy of the ICA-based portfolio choice models. In both cases the models will have to be derived for the particular case of these alternative ICA formulations.

A third avenue to be explored could be the improvement of ICA estimation methods and techniques. No single technique exists at present which allows for the testing and validation of the accuracy of the estimated components. Dedicated estimation methods for financial applications could help ICA become a more mainstream technique in financial economics, because its formulation has several interesting advantages. A last possible route for potential research would be the generalisation from a single period framework to a multi-period framework.

All in all the portfolio choice model developed and analysed in this thesis presents an interesting and first successful application of Independent Component Analysis to portfolio finance and simultaneously shows how an analytical higher moment portfolio choice model can be derived through its application. We are therefore convinced that this thesis could lead to future research along the lines suggested above, while having advanced the science in the field of portfolio choice models.

Appendix 1: *Expected Utility Theory*

A.1 Introduction

In the interest of bringing forth a complete argumentation of the chosen approach, we provide the reader with a brief overview of the evolution of utility theory including its advantages and weaknesses, so that the starting point for the work presented in this thesis is coherent and justified. The lesser importance of this chapter in the general context of the thesis has compelled us to relegate it to the appendix.

Classic microeconomic theory postulates that each agent pursues an optimising behaviour leading to a microeconomic equilibrium for the system as a whole. Each competitive firm knows the price at which it will sell the goods it will produce. In reality there is, however, significant uncertainty between the moment the decision to produce is made and the moment the goods are sold. Certain markets, as explained by Kaldor (1938) will see significant price fluctuations in between both points in time. Classic microeconomics did not regard this problem as significant.

Financial theory has always had great difficulty in dismissing the uncertainty inherent to decisions made today with regard to investments for the future. Instead, as postulated by Savage (1954), decisions made today are based on some probability beliefs, or subjective probability distributions, of which the investor has full knowledge. These distributions combined together as objective probabilities would allow the investor to address uncertainty as well as decisions using incomplete information.

Between probability belief and the actual investment decision lies the central element of this work, the investment decision criterion. It relates risk, uncertainty and profitability with beliefs and expectations for the future. The past 50 years have yielded several frameworks of which the maximisation of expected utility has remained the most prominent, since it considers the risk and profitability of an investment as one, defined by the probability beliefs referred to earlier. This generality has increased its popularity but has also posed mathematical challenges. To adequately deal with the probability beliefs and thus with the aspects of risk and profitability, are essential and represent the starting point of this work.

A.2 Risk, Uncertainty and Profitability

Mainstream literature often treats risk and uncertainty as synonyms and dissects them from the concept of profitability or return. The three concepts are certainly distinct, but they remain connected through the probability distribution. Knight (1921) defines risk as the awareness of both outcome and probability, while uncertainty is defined as having knowledge of the outcome, but the probabilities are unknown. Knowledge of risk therefore implies also having some knowledge about the profitability or returns.

Knight's distinction above is important in the context of investment criteria. An investor faced with a world devoid of risk will naturally choose whatever investment will yield the largest profit or return. This is in fact the premise of classical microeconomics, where price decisions are based on absolute knowledge of the future. Investment decisions are also based on absolute certainty. This is referred to as the Maximum Return Criterion (MRC). Any decision criterion is only applicable if it can be employed in a non-arbitrary manner, in that it will yield only one result. In the presence of risk and in contexts with more than one possible outcome, the MRC will lose its validity.

However, it is never the case that certainty prevails, and an investment yields only one possible return. In such instances, the investment criterion has to deal not only with risk, but also more generally, with uncertainty. Considering that the investor acts upon certain probability beliefs usually justifies the move from uncertainty to risk and this implies that each of the potential outcomes, however many there may be, can be associated with a subjective probability. This step is not trivial and is part of the assumptions generally made in financial theory which may be contested as we show in this thesis.

The normal way to address investment decisions in the case of risk is by considering the expected return, rather than the absolute return. Formally, this implies that rather than to speak in terms of returns, we speak in terms of the statistical expectation of returns. The decision criterion based on this thought is known as the Maximum Expected Return Criterion (MERC).

MERC provides an unambiguous ranking of all investment decisions based on the expected return of each. It is the logical consequence of including risk in the investment decision, and yet, as has been shown in the past, it can lead to paradoxes. The most prominent paradox is the St. Petersburg Paradox, which relates to the investment in a game of coin tosses. A coin is tossed until the first heads shows. At that point, the player is awarded a certain sum for each tails that showed previously.

In theory this game can be infinite. The question, which troubled Nikolaus Bernoulli and Gabriel Cramer, the two mathematicians who ultimately solved the problem, was the amount an investor is willing to invest in such a game. Put differently, what certain amount would be acceptable to an investor in order to be indifferent between playing and not playing, which is labelled the *certainty equivalent*. With the game as potentially infinite this amount should in theory also be infinite. If for example you pay £100 and after the first toss the first tails appears, one receives £1 for playing once and loses £99, as the first tails appeared right at the start. The *certainty equivalent* should therefore be a low amount, however, according to the MERC the expected prize of the game and therefore the *certainty equivalent*, should be infinite. If $\frac{1}{2^x}$ is the probability that the first tails shows up at the x^{th} toss, the certainty equivalent equals $\sum_{x=1}^{\infty}\frac{1}{2^x}2^{x-1}=\infty$.

The solution proposed by Bernoulli and Cramer (see (Levy & Sarnat, 1984) for more details) was to introduce the concept of the utility function. This function describes investor's preferences with respect to an investment, since the monetary value of money, as shown above, is often not a good indicator of preference. Important to the investor is the utility derived from the wealth rather than the wealth itself. The MERC is a criterion, which does not take the marginally reducing importance of wealth into account. When the St Petersburg Paradox is considered again, for an investor with for example a logarithmic utility function, the certainty equivalent now, $E[\log(W)] = \log\left(\sum_{x=1}^{\infty}\frac{1}{2^x}2^{x-1}\right) = \log(2)\sum_{x=1}^{\infty}\frac{x-1}{2^x} = \log(2)$, equals 2.

A.3 Maximum Expected Utility Criterion (MEUC)

Although Bernoulli and Cramer solved the St. Petersburg Paradox, an expected utility-based investment criterion only followed several centuries later with the development of the Borch (1974) and Borch (1990) whose original paper has been reprinted in various books, Ramsay (1931) and von Neuman and Morgenstern (1953). Their theory postulates that the alternative investments should be ranked following their expected utilities.

The ranking of the investments follows six *axioms*, which hold true for any investment under the assumption of the rational investor. These axioms are the following.

Axiom 1: comparability. For any two investments with a monetary outcome, the investor will always prefer one over the other.

Axiom 2: continuity. Given three investments A, B and C and given the investor prefers A over B and B over C, there will be a probability p, where the investor is indifferent between receiving B with certainty, or either A with probability p and or C with a probability $(1-p)$.

Axiom 3: interchangeability. Given the continuity axiom, if the investor is indifferent between two outcomes as described above and given one of these outcomes is part of an investment, then these two outcomes can be interchanged in an investment.

Axiom 4: transitivity. If the investor prefers A over B and B over C then the investor prefers A over C. This property also applies to the case where the investor is indifferent between the investments.

Axiom 5: decomposability. If the outcomes of an investment are other investments, the first "complex" investment can be decomposed into "simple" investments.

Axiom 6: monotonicity. In the case of certainty, this last axiom determines that the higher monetary outcome is always preferred.

The mathematical properties of each of the axioms will not be discussed here, however, they formulate a strict set of rules, which must apply for a utility function to be a valid choice. Through these axioms it can be shown that the MEUC is an optimal decision rule. If we have two arbitrary investments of which risk and return properties are known, it can be shown that one investment will indeed be preferred over another investment and that this preference corresponds to the investment with the highest utility.

Following the axioms, utility functions generally have two properties: they are an increasing function of wealth and they are concave. The first property is rather straightforward: more is better. The second one can be explained by the decreasing marginal utility of wealth, which implies that the marginal utility of an extra unit of wealth decreases as wealth increases.

The degree of curvature of the utility curve determines its risk aversion. However, utility curves are only defined up to affine transformations and therefore a measure of risk aversion is necessary which stays constant with respect to transformations. One such measure is the concept of risk aversion as defined by Pratt (1964) and Arrow (1965). It is often used in practice to distinguish between roughly two classes of risk aversion:

Absolute risk aversion (ARA): If we consider a utility function $U(W)$ of the W the wealth, the risk aversion function is defined as:

$$R(W) = \frac{-U''(W)}{U(W)} \tag{A.1.1}$$

Relative risk aversion (RRA): Is defined as equation (11.2) in which the wealth has been added as a factor.

$$R(W) = \frac{-WU''(W)}{U(W)} \tag{A.1.2}$$

The distinction between the two types of risk aversion resides in the dependence on overall wealth. A constant level or RRA implies a decreasing ARA, but the converse is not necessarily true.

Important examples of both types of utility functions are the following:

Exponential utility: This function belongs to the constant absolute risk averse family and is mathematically defined as:

$$U(W) = -\exp(-\theta W) \tag{A.1.3}$$

where θ is a coefficient.

Logarithmic utility: This function belongs to the family of constant relative risk averse functions. It is defined as:

$$U(W) = \log(W) \tag{A.1.4}$$

Power utility: this function is also often referred to as the isoelastic utility function and is defined as:

$$U(W) = \frac{W^{(1-\eta)} - 1}{1 - \eta} \tag{A.1.5}$$

Its relative risk aversion coefficient equals η.

All utility functions shown above are examples of constant relative or absolute risk aversion. Risk aversion can also be decreasing or increasing, which leads to positively or negatively skewed utility functions. Examples include the logarithmic utility, which exhibits DARA, and quadratic utility, which exhibits IARA.

Among these functions a particular subset is important in finance. They are those functions that also exhibit hyperbolic absolute risk aversion, or HARA. In these cases the risk aversion function is a hyperbolic. CARA and CRRA function belong to that subclass. Their importance stems first of all from their empirical validity, in particular the CRRA functions. In addition, HARA functions have the unique property of allowing a direct conversion between monetary and utility units.

The description given above gives only a quick overview of the MEUC. Several utility functions and subtleties of the presented functions are not described here. For a more detailed description of the framework, one can turn to Gerber and Pafumi (1998).

A.4 Arguments Against the MEUC

Expected utility theory as presented above has encountered criticism over the years, placing doubt next to its validity. We do not intend to give a full account of the arguments made against it, but merely to show which general conclusions were drawn and on what they were based.

The main source of criticism stems from laboratory findings that seem to contradict the basic assumptions of expected utility theory, or highlight paradoxes in their findings. The primary starting point of all of these experiments is the doubt about the rationality hypothesis of economic agents.

Five findings in particular are worth mentioning. Classical economic theory prescribes that utility functions should be smooth in their domain. Experimental results by Kahneman (2003) and others indicate that economic agents tend to have a reference point to which they refer when making investment decisions. This reference point divides the utility function into two parts, with different degrees of risk aversion because the aversion to gain and loss are not identical.

Thaler's endowment effect presented in Kahneman, Knetsch and Thaler (1990) further confirms this finding by showing that the aversion to loss is far greater than the aversion to gain. In practice, however, this effect does not necessarily have to discredit expected utility theory in general as smooth utility functions could simply be a particular case of the more general kinked utility functions. That hypothesis does not hold, however, when the economic agent has a fixed reference point that does not move with gains from investments.

Similar to the endowment effect is first-order risk aversion, or loss aversion, as described by Rabin (2000). An agent with a CARA utility function and a risk aversion coefficient of 0.001, presented with a 50:50 gamble of losing 1000 or gaining an infinite amount will always reject this gamble according to Rabin's findings. Rabin argues for the existence of a calibration theorem, which implies that experimental data from laboratory experiments are used to calibrate utility curves as it is evident that expected utility cannot predict the economic behaviour of agents. This theorem can, however, be used both ways.

Rabin and Thaler (2001) further developed their argument by considering several examples of reasonable gambles that expected utility maximisers would reject. They consider a gamble with a 50:50 chance of winning 11, or losing 10. Such a gamble would indeed be rejected. Given decreasing marginal utility, this finding then leads to the conclusion that bigger bets with a 50:50 chance of winning an infinite amount while losing 100 would also be rejected. This argument does indeed question the validity of expected utility maximisation.

An additional paradox which attracts our attention, is the Allais paradox by Allais (1953). Allais' paradox relates to the fact that expected utilities are linear in their probabilities. This implies that the marginal utility of a 1% increase in the probability of gaining or losing something will be equal no matter what the starting probability was. Cases can be constructed in which equal shifts in probabilities between several lotteries should produce equal shifts in expected utilities, but where in practice, however, this operation leads to completely different actual preferences on the part of the economic agent.

In light of these paradoxes and inconsistencies with laboratory findings, several alternatives to expected utility have been proposed. The most prominent example is prospect theory introduced by Kahneman and Tversky (1979). The theory builds around a reference point, the centre point of the agent's decision making. As agents generally have a different aversion to a loss than to a gain, the valuation function, which describes the agent's value of gains and losses, has a different shape on both sides of the reference point. Additionally, a weight function is defined which applies a weighting to the different outcomes, as the valuation function is defined with respect to the deviation from the reference and not with respect to the outcome itself.

The consequence of this approach is that the preferences as expressed by the weighted valuation function are no longer linear in their probabilities. This resolves of the paradox arising from linear expected utilities. The structure of the functions expresses gains and losses as a deviation with respect to the reference point and not as an absolute final gain or loss. Kahneman and Tversky (1979) show the magnitude of the gain or loss is less important than deviation from the starting point.

Theories of lesser prominence are the Regret theory of Loomes and Sugden (1982), Rank-dependent utility by Quiggin (1982). As with prospect theory, both theories aim to mediate the Allais paradox. The Regret theory models choice under uncertainty as the minimisation of a function of the regret vector, which is defined as the difference between the best possible outcome and the outcome of a given choice.

Rank-dependent expected utility on the other hand generalises expected utility theory by overweighting extreme and low probability events. In doing so, the inconsistency highlighted by the Allais paradox is addressed. The approach was later also added to the original prospect theory.

A.5 Arguments in favour of MEUC

Several practical and conceptual experiments have been put forward over the years to undermine the validity of the MEUC as a model to describe the behaviour of economic agents when faced with choice under uncertainty. It led to the very successful field of behavioural economics, where focus is on the behaviour of the individual economic agent in a verifiable context, such as a laboratory, for example.

In describing the behaviour of the investor, or indeed the market, the direct link between laboratory results and observable behaviour has always been less clear. Prominent in the explanation of this fact was a theory invoking the notion of Dutch Books, as proposed by Yaari (1985).

The theory suggests that if the behaviour of investors had in fact been in line with the results observed in the laboratory and if indeed the MEUC was a faulty model for describing what decision tools investors were actually using, an arbitrage opportunity would have surely been created for smart investors to use to good advantage. Such opportunities are referred to as Dutch Books, because the decision under uncertainty is based on a faulty model that would have always led to a gain for the investor who takes advantage of this fact.

Watt (2002) provides a more direct response to the arguments of Rabin and Thaler (2001). The examples presented by Rabin and Thaler (2001) may indeed be correct in principle, however, the deductions and conclusion drawn from them are not. The two gambles presented by Rabin and Thaler (2001) cannot be linked through the axiom of decreasing marginal utility without words of caution. In fact the second gamble makes perfect sense for any expected utility maximiser.

Watt (2002) gives an example to show that an economic agent with logarithmic utility presented with the gambles mentioned above will indeed be taken, given a certain level of initial wealth. When presented with a gamble with 50:50 chance of winning 11 or losing 10, the initial wealth works out to be minimally 110. Taking this conclusion and considering the 50:50 gamble of winning an amount X and losing 100, the initial wealth required must be greater than 1211,11. This amount by itself is not at all odd, given the expected win and loss of the gamble. Consequently, in deducing that high stakes bets would be turned down on the basis of small stakes bets, implies the presence of high degrees of risk aversion, which Rabin and Thaler (2001) had not taken into account. When correctly applied the MEUC criterion predicts that the expected utility maximiser will take the gambles, as one would logically expect.

A.6 Conclusion

The main aim of this appendix has been to present an objective review of the point of departure for most investment models of recent history: the maximisation of expected utility. This review is highly significant, as it should clearly show the benefits as well as the drawbacks of expected utility theory in the context of financial decision-making.

The examples given above and the fact that the MEUC is still very widely applied in industry and in academia, allows us to conclude that its framework for decision making under uncertainty remains the benchmark framework. Though the evidence against the use of MEUC has become more voluminous over the years, the lack of consensus regarding true behaviour of the economic agent outside the laboratory and inside a market context have left somewhat of a gap to be filled by an alternative and more accurate theory.

Until that theory is identified, the MEUC will remain as the main reference point for the modelling of decision-making under uncertainty. Many of the most popular and most debated financial models still find their theoretical basis in expected utility theory. Most prominently among the models is the theory related to portfolio choice, which has been the main focus of this thesis. In light of the arguments and discussion elaborated in these pages on a theory, which has over half a decade of research to its credit, we conclude that expected utility theory is a solid starting point for financial decision-making models.

Appendix 2:
Diversification Delta[5]

[5] This chapter is entirely based on and a partial transcript of a paper prepared and published by the author of this thesis with the following reference: Vermorken, Medda and Schröder (2012)

A.1 Introduction

Over the years, several methods have been proposed to increase the accuracy of diversification measurement. In this appendix the aim is to introduce a new measure of diversification referred to as the Diversification Delta (DD), which retains the ease of application and interpretation of the common measures of diversification, the variance and the correlation, but extends it to include all higher moments of the returns distribution of the considered assets. At the core of this new measure is the concept of Shannon entropy, otherwise known as information entropy.

Shannon entropy can measure the uncertainty related to the entire statistical distribution and by so doing, the Diversification Delta is a response to Samuelson's criticism. Entropy captures the reduction in uncertainty as the portfolio of stocks becomes more diversified, in other words, increased diversification of a portfolio reduces uncertainty and lowers entropy in its final outcome.

Differential entropy, i.e., the continuous generalisation of Shannon entropy (1948), is a measure of uncertainty of a random variable and it is the concept, which underpins the Diversification Delta. In our context, differential entropy represents the investor's average uncertainty of the returns of an investment. Where variance quantifies the concentration of a return distribution around its mean, entropy measures concentration irrespective of its location in the distribution. For example, high levels of concentration around the tails of a distribution of asset returns will affect the entropy of the distribution, whereas variance will remain largely unaffected.

A.2 Derivation

The entropy H of a random variable X with possible values $x \in \mathcal{X}$, $\mathcal{X} = \mathbb{R}$, can mathematically be defined as:

$$H = -\int_{\mathcal{X}} f(x) \, log \, f(x) dx \tag{A.2.1}$$

where $f(x)$ is the probability density function of X. H is formulated as the expectation of the natural logarithm of the probability density function. Whenever areas of concentration occur and some outcomes are more likely than others, $f(x)$ increases as a consequence. Such areas of concentration decrease the uncertainty of the possible outcomes of a random draw. The entropy will therefore decrease when a distribution is more concentrated around a certain point.

As it can be observed from equation (A.2.1), is exclusively related to probabilities in the discrete case and to the distribution functions in the continuous case, making complete abstraction of the nominal nature of the associated random variable. Two distinct probability distributions could therefore exist with the same entropy but with different variances. However, empirical financial data belong to a sub-class of single-maximum with monotonicity on both sides of the maximum, (see Cont (2001) for a review of the properties of financial data). In practice the above-mentioned case is not observed, as it would require perfectly symmetrical anomalies in the distribution of returns.

At this stage we can introduce the formal definition of the Diversification Delta. Let $X_1, X_2, \ldots, X_n$ be risky assets of universe U. P is a portfolio with portfolio weights

$P = (\alpha_1, \dots, \alpha_n)$ *and* $\sum_{i=1}^{n} \alpha_i = 1$. In order to facilitate its interpretation we define the Diversification Delta $\mathrm{DD(P)}$ as the following ratio:

$$DD(P) = \frac{exp\left(\sum_{i=1}^{n} \alpha_i H(X_i)\right) - exp\left(H\left(\sum_{i=1}^{n} \alpha_i X_i\right)\right)}{exp\left(\sum_{i=1}^{n} \alpha_i H(X_i)\right)} = \frac{exp\,\overline{(H(X))} - exp\,(H(P))}{exp\,\left(\overline{H(X)}\right)} \; with\, i = 1, \dots, n \qquad \text{(A.2.2)}$$

DD*(P)* is the ratio of the weighted average entropy of the assets $\overline{H(X)}$, minus the entropy of the portfolio $H(P)$, divided by the weighted average entropy of the assets $\overline{H(X)}$. The ratio measures the relative reduction in entropy, or put differently, by combining into a portfolio assets X_i with weights α_i, the ratio evaluates the relative reduction in uncertainty. Campbell (1966) shows how the exponential value of entropy retains the sought after characteristics of the measure while allowing us to avoid the case of a singularity when $\overline{H(X)} = 0$.

The interpretation of the DD is rather straightforward. The DD is defined as a ratio varying between zero and one. A value of one indicates that only market risk remains in the portfolio and all idiosyncratic risk has been diversified. In such a case there is no longer any difference between the weighted average of the entropy of the individual assets and the entropy of the portfolio as a whole. This means that the weighted specific risk of the single stocks has no influence over the portfolio and what remains is the risk common to all, that is, the market risk.

In order to illustrate the DD concept, it is interesting to consider it in the context of an equally weighted portfolio of standardised gaussian data; these data can be generated easily. The use of standardised data allows us to make abstraction of the effects of variance and we can most effectively understand how diversification affects the DD. We analyse two cases.

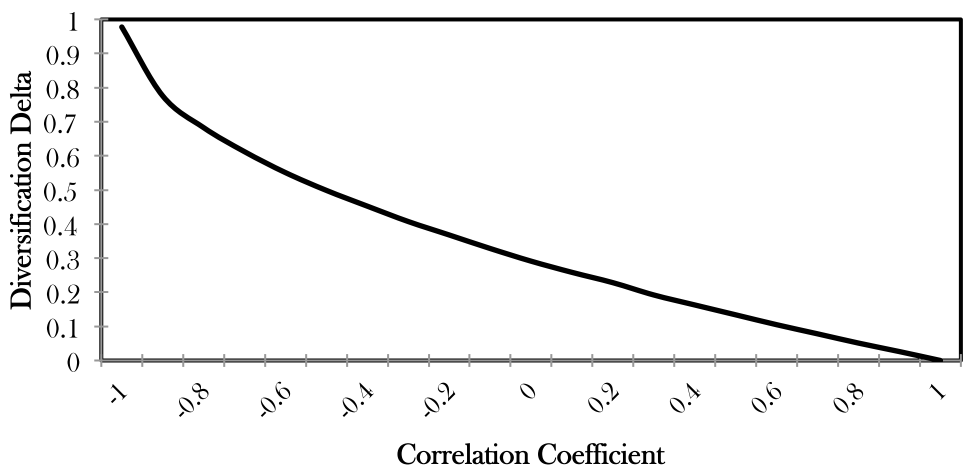
1. The DD and its relation to Pearson's correlation coefficient using an equally weighted two-asset portfolio X_1, X_2.
2. The DD when the portfolio's size increases from 1 to n assets in order to understand the effect of the pool size on the measure (Elton and Gruber, 1997).

For the estimation of the DD and the entropy, we use a recently developed non-parametric method known as the k-d-Partitioning by Stowell and Plumbley (2009).

Case 1: Schematic overview using an equally weighted two-asset portfolio X_1, X_2

We generate two sets of random data with an ex-ante determined correlation coefficient varying between -1 and 1. The data are standardised gaussian data. We subsequently compute the DD for each chosen value of the correlation coefficient and repeat this procedure for 1000 iterations in order to avoid a selection bias. The result, averaged across the 1000 iterations, is plotted in Figure A.1.

Figure A-1: DD and Pearson's Correlation Coefficient



The relationship between the DD and the correlation coefficient is non-linear as expected. The definition of the DD, which through entropy is non-linear, and the correlation coefficient, which is a linear dependence measure, leads to this non-linearity. It is noteworthy that the decrease in the DD with respect to the correlation coefficient is more accentuated for negatively correlated stocks. The implication here is that, where portfolio selection or portfolio management applications are concerned, the DD will indicate a favourable bias towards uncorrelated stocks.

An overview of the various cases and conclusions that can be drawn from Case 1 is presented in Figure A.2.

Figure A-2: Overview of cases

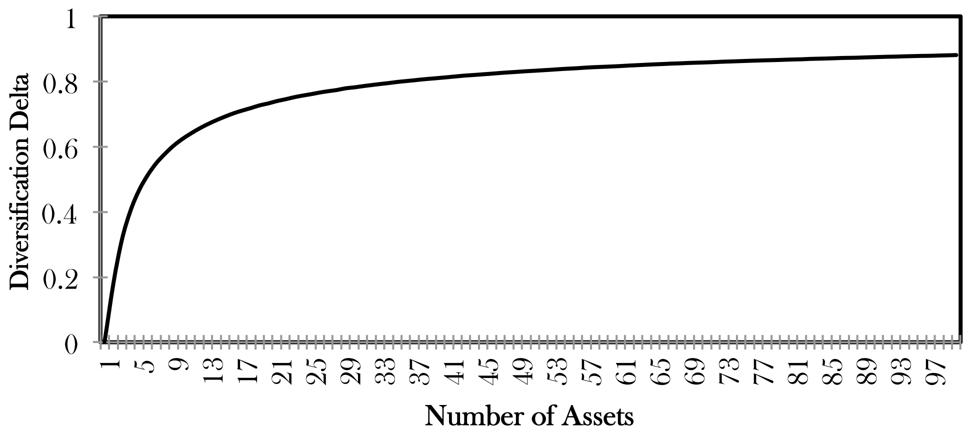
Linear Correlation	Diversification Delta	Portfolio Variance	Weighted Average Entropy of Assets		Portfolio Entropy	Distribution of Returns
-1	DD ≤ 1	Minimum	Hx	≥	Hp = 0	If return distributions of the two considered assets are equal and portfolio weights are equal, weighted sum of asset returns will be zero, Hp equals zero, diversification will reach its maximal value.
1	DD ≥ 0	Undetermined	Hx	=	Hp	If returns are equally distributed and perfectly correlated, the weighted sum of returns will be exactly similar to the individual returns. Hx equals Hp and diversification is minimal.
0	0 ≤ DD ≤ 1	Reduced	Hx	≥	Hp	If returns are equally distributed and uncorrelated, the weighted sum of returns will reduce the risk associated with the two assets. Hx will be larger than Hp and some diversification is achieved.
[-1,1]	0 ≤ DD ≤ 1	Reduced	Hx	≥	Hp	If returns are distributed according to different distributions, with correlations varying between minus one and plus one, the combination in a portfolio of assets can reduce the uncertainty associated with returns by increasing their concentration.

Case 2: Increasing the pool size and the effect on the DD

Our second case focuses on the most basic form of diversification, spreading wealth across assets. By following Elton and Gruber (1997) we use a set of n assets and systematically increase the portfolio's size from 1 to n, monitoring the effect on the DD. In order to avoid a selection bias, we generate 1000 sets of standardized gaussian data, with each set consisting of 100 series. Each series contains randomly generated observations that follow a gaussian distribution. The DD is measured for each set as n increases from 1 to 100. The result plotted in Figure A.3 depicts an average across the 1000 sets.

Consistent with the findings of Elton and Gruber, the DD makes most of its gains with the first 30 assets. After this point the entropy of the portfolio remains largely constant, while the average entropy of the individual assets could increase further. Each new asset adds new specific risk to the mix, which, due to the large number of assets, is averaged in the calculation of the portfolio entropy. The DD therefore increases slowly towards the value of 1, as all idiosyncratic risk is systematically eliminated and only market risk remains.

Figure A-3: The Dependence of the DD on Portfolio Size



A.3 Conclusion

We can conclude by underlining the three distinct advantages of the DD. Firstly, it is a non-parametric measure of diversification and includes higher moments of the distribution, thereby addressing Samuelson's criticism. Secondly, it is related to both correlation and variance; and thirdly, the DD is easily interpretable due to its mathematical formulation.

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